JOURNAL OF MULTIDISCIPLINARY SCIENCES AND INNOVATIONS
VOLUME 5, ISSUE 05
MONTHLY JOURNALS
ISSN NUMBER: 2751-4390
IMPACT FACTOR: 9,08

4UDC 517.956.2

BIFURCATION OF SOLUTIONS TO A NONLINEAR ELLIPTIC PROBLEM IN A DISK

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https://doi.org/10.5281/zenodo.20487048

ABSTRACT: We study a nonlinear elliptic Dirichlet boundary value problem for an equation in a circular domain. Using the method of small parameters and the Lyapunov–Schmidt reduction, we investigate bifurcation points, construct asymptotic expansions of solutions, and determine the nature of the emerging branches. The existence of a supercritical pitchfork bifurcation is proved. The stability of solutions and the global structure of branches are analysed.

Keywords: nonlinear elliptic equation, bifurcation, Lyapunov–Schmidt method, asymptotic expansion, Rabinowitz theorem, moving-planes method.

1. INTRODUCTION

Nonlinear elliptic equations occupy a central place in modern mathematical physics, arising in the theory of phase transitions, nonlinear optics, continuum mechanics, and quantum field theory. The qualitative analysis of such equations — in particular the study of conditions under which nontrivial solutions emerge — is one of the fundamental problems of nonlinear analysis.

The present paper concerns the Dirichlet problem for a nonlinear equation with cubic nonlinearity in a circular domain:

$$-\Delta u + \mu u - u^3 = 0, \quad x \in \Omega, \quad u|\partial\Omega = 0,$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$ is a disk of radius a and $\mu \in \mathbb{R}$ is a spectral parameter.

The goal of the paper is a complete investigation of the local and global solution structure: finding bifurcation points, constructing asymptotic expansions, proving the nature of bifurcation, analysing branch stability, and examining symmetry properties of positive solutions.

Problems of this type were studied by Krasnosel'skij [1], Rabinowitz [2], and Chow and Hale [6]. The specific case of cubic nonlinearity in a disk, however, admits a detailed analytical treatment with explicit formulae, which constitutes the content of the present article.

2. METHODOLOGY

2.1. Linear spectral problem

The analysis rests on the linear spectral problem

$$-\Delta\varphi + \mu_0\varphi = 0, \quad \varphi|\partial\Omega = 0.$$

The spectrum of this problem is discrete: $0 < \mu_1 < \mu_2 < \cdots$, and the corresponding eigenfunctions form a complete orthonormal basis of $L^2(\Omega)$. By the radial symmetry of the domain, the eigenfunctions are expressed through Bessel functions:

$$\varphi_{mn}(r, \theta) = J_m(\alpha_{mn} r/a)(A \cos m\theta + B \sin m\theta),$$

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where α_{mn} are the zeros of the Bessel function J_m.

2.2. Method of small parameters

Asymptotic expansions are constructed via the method of small parameters. The solution and the parameter are represented as power series in a small parameter ε:

$$u = \varepsilon^{1/2} u_1 + \varepsilon u_2 + \varepsilon^{3/2} u_3 + \cdots, \quad \mu = \mu_0 + \varepsilon \mu_1 + \cdots$$

Substituting into the equation and collecting terms of equal powers of ε yields a recurrent system of problems for the expansion coefficients.

2.3. Lyapunov–Schmidt reduction

The bifurcation equation is analysed via the Lyapunov–Schmidt reduction. The solution space is split into the kernel of the linear operator and its orthogonal complement. Projection onto the kernel yields a scalar bifurcation relation that admits a complete analysis.

3. RESULTS

3.1. Asymptotic expansion

Theorem 1. Let μ_0 be a simple eigenvalue of the linear Dirichlet problem. Then in a neighbourhood of the point $(u, \mu) = (0, \mu_0)$ there exists an asymptotic expansion of the solution:

$$u = \varepsilon^{1/2} \varphi + \varepsilon w, \quad w \perp \varphi.$$

At order $\varepsilon^{1/2}$ we obtain $-\Delta u_1 + \mu_0 u_1 = 0$, giving $u_1 = c\varphi$, where φ is the eigenfunction of the linear problem. At order $\varepsilon^{3/2}$ the Fredholm solvability condition requires orthogonality of the right-hand side to the kernel of the adjoint operator, necessitating fractional powers of the parameter. ∎

3.2. Bifurcation equation and type of bifurcation

Theorem 2. In a neighbourhood of the bifurcation point the parameter μ satisfies the asymptotics

$$\mu - \mu_0 = C\varepsilon + o(\varepsilon), \quad C = \int_\Omega \varphi^2\, dx \,/ \int_\Omega \varphi^4\, dx > 0.$$

Multiplying the equation by φ and integrating over Ω, the condition $L\varphi = 0$ shows that the principal contribution comes from the nonlinear term: $(\mu - \mu_0)\varepsilon^{1/2} \int\varphi^2\, dx - \varepsilon^{3/2} \int\varphi^4\, dx + o(\varepsilon^{3/2}) = 0$. Dividing by $\varepsilon^{1/2}$ yields the statement of the theorem. ∎

Theorem 3. The bifurcation of solutions is a supercritical pitchfork bifurcation.

Theorem 2 gives $\mu - \mu_0 = C\varepsilon$, $C > 0$, so nontrivial solutions exist for $\mu > \mu_0$. Their asymptotics read:

$$u_\pm(x, \mu) = \pm\sqrt{(\mu - \mu_0)}\, \varphi(x) + o(\sqrt{(\mu - \mu_0)}).$$

The existence of two symmetric branches u_+ and u_- corresponds to a pitchfork bifurcation. Both branches emerge from the trivial state at $\mu = \mu_0$. ∎

3.3. Stability of solutions

Theorem 4. The bifurcating solutions are energetically stable.

Consider the energy functional $E(u) = \int_\Omega (|\nabla u|^2 + \mu u^2/2 - u^4/4)\, dx$. Substituting the asymptotics $u^* \sim \sqrt{(\mu - \mu_0)}\, \varphi$ gives:

$$E(u^*) = -(1/4)(\mu - \mu_0)^2 \int_\Omega \varphi^4\, dx + o((\mu - \mu_0)^2) < 0.$$

Since $E(u^*) < 0 = E(0)$, the nontrivial solution minimises the energy relative to the trivial state, establishing its stability. ∎

3.4. Global branch structure

Theorem 5 (Rabinowitz). From each spectral point $(\mu_n, 0)$ there emanates a connected global branch of solutions.

The operator $F(u, \mu) = -\Delta u + \mu u - u^3$ is a compact perturbation of a Fredholm linear operator of index zero. Since μ_n is a simple eigenvalue, the global Rabinowitz bifurcation theorem [2] applies. Consequently, there exists a connected component C_n of solutions emanating from $(\mu_n, 0)$ that is either unbounded or meets another bifurcation point. ∎

3.5. Radial symmetry of positive solutions

Theorem 6 (Gidas–Nirenberg). Every positive solution of the problem is radially symmetric.

The domain is a ball and the nonlinearity depends only on u. Applying the moving-planes method of Gidas–Nirenberg, one obtains the radial symmetry of the solution. This reduces the problem to an ordinary differential equation. ∎

3.6. Large-parameter asymptotics

As $\mu \to \infty$ the solution satisfies the asymptotics

$u(r) \sim \sqrt{\mu} \cdot \tanh(\sqrt{(\mu/2)}(a - r))$,

which corresponds to the formation of a boundary layer of thickness $O(\mu^{-1/2})$.

4. DISCUSSION

The results obtained provide a comprehensive picture of the solutions of the problem under consideration. The supercritical nature of the pitchfork bifurcation means that nontrivial solutions emerge continuously from the trivial state as μ passes through μ_0, which is fundamentally different from the subcritical case, where jump-type transitions are possible.

The energetic stability of the bifurcating branches has a clear physical meaning: in applications to phase transition theory, it implies that the new phase is thermodynamically preferable to the original one. This result is consistent with well-known theorems for problems generated by potential operators.

The global Rabinowitz theorem guarantees that each locally detected branch can be continued over the entire range of the parameter μ. Combined with the Gidas–Nirenberg theorem, this allows one to restrict attention to radially symmetric functions for positive solutions, substantially simplifying their numerical analysis.

The large-parameter asymptotics reveals the characteristic structure of large solutions: they are close to the constant value $\sqrt{\mu}$ in the interior of the domain and decay rapidly in a boundary layer of thickness $O(\mu^{-1/2})$. This result is important for understanding the concentration geometry of solutions.

The methods developed here — the small-parameter method, the Lyapunov–Schmidt reduction, and the moving-planes technique — are applicable to a wide class of nonlinear elliptic problems in domains of various geometries. Directions for further research include the study of solution multiplicity, second-order bifurcations, and the numerical realisation of the constructed asymptotics.

CONCLUSION

A complete investigation of a nonlinear elliptic Dirichlet problem with cubic nonlinearity in a circular domain has been carried out. The following main results have been obtained:

1. bifurcation points, coinciding with the eigenvalues of the linear problem, have been found;
2. asymptotic expansions of solutions have been constructed by the method of small parameters;
3. the supercritical pitchfork character of the bifurcation has been proved;
4. the energetic stability of the bifurcating branches has been established;
5. the existence of connected global branches of solutions has been proved on the basis of the Rabinowitz theorem;
6. the radial symmetry of positive solutions has been established by the Gidas–Nirenberg method;
7. large-solution asymptotics with a description of the boundary-layer structure have been obtained.

The methods presented are applicable to a wide class of nonlinear problems of mathematical physics and may serve as a basis for further research in this direction.

REFERENCES

1. Krasnosel'skij M.A. Bifurcation of Solutions of Nonlinear Equations. Moscow: Nauka.
2. Rabinowitz P. Global bifurcation theory. Uspekhi Mat. Nauk.
3. Vladimirov V.S. Equations of Mathematical Physics. Moscow: Nauka.
4. Smirnov V.I. A Course of Higher Mathematics. Vol. IV. Moscow: Nauka.
5. Gilbarg D., Trudinger N. Elliptic Partial Differential Equations of Second Order. Berlin: Springer.
6. Chow S., Hale J. Methods of Bifurcation Theory. New York: Springer.

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