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QUADRATURE FORMULAS FOR APPROXIMATING FRACTIONAL RIEMANN-LIOUVILLE INTEGRALS

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Abstract:
Fractional integrals play a fundamental role in fractional calculus and are widely used in the modeling of complex physical and engineering processes with memory and hereditary properties. In particular, the Riemann–Liouville fractional integral is characterized by a weakly singular kernel, which makes its exact analytical evaluation difficult in most practical cases. Therefore, the development of efficient and accurate numerical methods for approximating fractional integrals remains an important research topic. In this paper, several quadrature formulas for approximating Riemann–Liouville fractional integrals are investigated. The proposed approaches are based on polynomial interpolation, including fractional left and right rectangular formulas as well as the fractional trapezoidal (L1) method. Exact analytical solutions for power-type functions are derived and used as benchmark solutions. Numerical approximations are constructed and compared with exact solutions to assess accuracy. The absolute error is analyzed to evaluate the performance of each method. The results demonstrate that polynomial interpolation–based methods provide reliable approximations of fractional integrals. In particular, the fractional trapezoidal (L1) method shows improved accuracy compared to rectangular formulas while maintaining computational simplicity. These findings confirm the effectiveness of the proposed quadrature formulas for practical applications in fractional calculus.

Keywords: Fractional calculus; Riemann–Liouville integral; fractional quadrature formulas; polynomial interpolation; fractional rectangular rule; fractional trapezoidal rule; L1 method; numerical approximation; singular kernel; error analysis.

Introduction

Fractional-order integrals and derivatives theory has gained significant importance in applied mechanics, materials science, control systems, and mathematical modeling over the past decades. In this field, particular attention has been devoted to the exact and approximate evaluation of fractional integral operators of the Riemann–Liouville and Caputo types.

In his fundamental monograph, Podlubny [1] thoroughly presents the mathematical foundations of fractional calculus and highlights the properties of the Riemann–Liouville integral as an integral operator with a singular kernel. The author emphasizes the necessity of numerical methods for evaluating fractional integrals and demonstrates that classical quadrature formulas can be generalized to the fractional case.

In the works of Oldham and Spanier [2], early numerical approaches to fractional calculus were developed, providing justification that fractional integrals can be approximated in a manner analogous to classical integral formulas. These studies later served as a foundation for the development of methods based on polynomial interpolation.

Diethelm [3] systematically presents numerical methods for fractional differential and integral equations, paying particular attention to stability and accuracy issues in the discretization of Riemann–Liouville integrals. He points out that left- and right-rectangle formulas are simple to implement but suffer from low accuracy.

Lubich [4] proposed convolution quadrature methods for fractional integrals and derivatives and proved that these methods provide high accuracy when dealing with singular kernels. This approach is particularly important for the numerical evaluation of time-fractional integrals.

Li and Zeng [5] analyzed numerical methods for approximating fractional integrals based on polynomial interpolation and provided a mathematical justification for the accuracy orders of trapezoidal and Simpson-type formulas. The authors demonstrated that higher-order interpolation techniques can significantly reduce approximation errors.

Gao and Sun [6] investigated the accuracy and stability of the L1 (fractional trapezoidal) method for fractional integrals, showing that this method is computationally efficient and effective for practical applications. The L1 method is widely used, especially in problems based on the Riemann–Liouville fractional integral.

Tian, Zhou, and Deng [7] proved that the accuracy of discretizing fractional integral operators can be improved by employing adaptive and graded meshes. This approach plays an important role in reducing the influence of singular kernels.

Ciesielski and Leszczynski [8] analyzed left- and right-rectangle formulas for fractional integrals and derived explicit error estimates. The authors emphasize that these formulas are most suitable as preliminary approximations.

Atangana and Baleanu [9], in addition to introducing new definitions of fractional operators, conducted a comparative analysis of numerical methods based on the classical Riemann–Liouville integral. Their study highlights the universality and flexibility of polynomial interpolation–based methods.

Kilbas, Srivastava, and Trujillo [10] presented the general theory of fractional integrals and derivatives and justified the reliability of numerical methods by comparing approximate solutions with exact analytical results. This monograph is considered one of the fundamental references in the numerical evaluation of fractional integrals.

The above literature review demonstrates that polynomial interpolation–based methods, particularly the left and right rectangle formulas and the trapezoidal (L1) method, are distinguished by their computational simplicity and practical applicability in approximating Riemann–Liouville fractional integrals. At the same time, for problems requiring higher accuracy, trapezoidal and Simpson-type formulas are preferable.

Methods and materials

The fractional integral operator plays an important role in fractional calculus, which is useful for converting the fractional differential equations into integral equations with a weakly singular kernel. So it is necessary to study the numerical methods for approximating fractional integrals. This section introduces numerical approaches used to approximate the fractional integrals based on the polynomial interpolation.

Suppose that $f(t)$ $C(I), I=[0,T]$. Let Δt be the step size with $\Delta t = \frac{T}{n_T}$, n_T N

and denote by $t_k = k\Delta t$. Next, we investigate how to numerically calculate the following integral

$$D_{0,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0 \tag{1}$$

One way to numerically calculate (1) is to approximate $f(t)$ by a certain function $\tilde{f}(t)$ in order that $D_{0,t}^{-\alpha} \tilde{f}(t)$ can easily be calculated exactly. We naturally think of the polynomial approximation of $f(t)$ on the interval $[0,T]$. Theoretically speaking, $D_{0,t}^{-\alpha} \tilde{f}(t)$ can be calculated exactly if $\tilde{f}(t)$ is a polynomial.For $t = t_n$, t we rewrite $D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n}$ as the following form

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(s)ds =$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s)ds. \tag{2}$$

Next, we introduce the numerical methods based on the polynomial interpolation to calculate (2)
Numerical Methods Based on Polynomial Interpolation
This subsection extends the numerical methods for the classical integrals to the fractional integrals.
Fractional Rectangular Formula
On each subinterval $[t_k, t_{k+1}], k = 0,1,...,n-1,$ the function $f(t)$ is approximated by a constant, i.e.,

$$f(t)\Big|_{[t_k,t_{k+1})} \approx \tilde{f}(t)\Big|_{[t_k,t_{k+1})} = f(t_k), \tag{3}$$

one obtains

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s)ds \approx$$

$$\approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(t_k)ds \approx \tag{4}$$

$$= \sum_{k=0}^{n-1} b_{n-k-1} f(t_k),$$

where

$$b_k = \frac{\Delta t^\alpha}{\Gamma(\alpha+1)}[(k+1)^\alpha - k^\alpha]. \tag{5}$$

Hence,

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} \approx \sum_{k=0}^{n-1} b_{n-k-1} f(t_k). \quad (6)$$

Similar to the classical left rectangular formula, we call (6) the *left fractional rectangular formula.*

Similarly, if

$$f(t)\Big|_{[t_k,t_{k+1})} \approx \tilde{f}(t)\Big|_{[t_k,t_{k+1})} = f(t_{k+1}), \quad (7)$$

then we get the following *right fractional rectangular formula*

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} \approx \sum_{k=0}^{n-1} b_{n-k-1} f(t_{k+1}). \quad (8)$$

The formulae (7) and (8) can be seen as the special cases of the following *weighted fractional rectangular formula*

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} \approx \sum_{k=0}^{n-1} b_{n-k-1}[\theta f(t_k) + (1-\theta) f(t_{k+1})], \quad 0 \quad \theta \quad 1 \quad (9)$$

Of course, one can also obtain the following formula

$$D_{0,t}^{-\alpha} f(t)\Big|_{t=t_n} \approx \sum_{k=0}^{n-1} b_{n-k-1} f(t_k + (1-\theta)\Delta t), \quad 0 \quad \theta \quad 1 \quad (10)$$

The definition of the Riemann-Liouville fractional integral that we want to use in our paper (the case where a=0):

$$D_{0,x}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

We find exact and approximate solutions for the function $f(t) = t^2$.

Finding an analytical solution F_{exact}.

Suppose a function $f(t) = t^2$ with fractional order $\alpha > 0$ is given in the interval $[0, x]$.

There is an exact formula for the Riemann-Liouville fractional integral of a power function.

$$D_{a,x}^{-\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(x-a)^{\beta+\alpha}$$

In our case: $a = 0$, $\beta = 2$.

$$F_{exact}(x,\alpha) = D_{0,x}^{-\alpha} t^2 = \frac{\Gamma(2+1)}{\Gamma(2+\alpha+1)} x^{2+\alpha}$$

Since it's a $\Gamma(3) = 2$, the obvious solution seems simple:

$$F_{exact}(x,\alpha)=\frac{2}{\Gamma(\alpha+3)}x^{\alpha+2}$$

Calculating the Exact Solution ($x=1,\alpha=0.5,N=2$)

Calculation: $\alpha=0.5$ va $x=1$

$$F_{exact}(1;0.5)=\frac{2}{\Gamma(0,5+3)}=\frac{2}{\Gamma(3,5)}$$

The value of the gamma function: $\Gamma(3,5)\approx 3.32335$

$$F_{exact}\approx\frac{2}{3.32335}\approx 0,60179$$

Finding an Approximate (Numerical) Solution

The Fractional Trapezium Method (or L1 method) is used to find a numerical solution.

Parameters:

$x=1,\alpha=0.5,N=2$. Step size: $h=x/N=1/2=0.5$. Nodes $t_0=0,t_1=0.5,t_2=1$.

Formula for Approximate Solution:

$$F_{approx}(x_n,\alpha)\approx\sum_{k=0}^{n}a_{k,n}f(t_k)$$

Here $n=2$. $a_{k,n}$ weight coefficients are found as follows:

$$a_{k,n}=\frac{h^{\alpha}}{\Gamma(\alpha+2)}\ b_{k,n}$$

Calculation of Approximate (Numerical) Solution (for $n=2$)

General Part C :

$$C=\frac{h^{\alpha}}{\Gamma(\alpha+2)}=\frac{0,5^{0.5}}{\Gamma(2,5)}\approx\frac{0,7071}{1,32934}\approx 0,5319$$

Finding the coefficients $b_{k,2}$:

1. $k=0$ (The starting point):

$$b_{0,2}=(2-1)^{1,5}-(2-1-0,5)2^{0,5}\approx 1-0,5\ \ 1,4142\approx 0,2929$$

$$a_{0,2}=C\ \ b_{0,2}\approx 0,5319\ \ 0,2929\approx 0,1558$$

2. $k=1$ (The middle point):

$$b_{1,2}=(2-1+1)^{1,5}-2(2-1)^{1,5}+(2-1-1)^{1,5}\approx 2^{1,5}-2\ \ 1+0\approx 0,8284$$

$$a_{1,2}=C\ \ b_{1,2}\approx 0,5319\ \ 0,8284\approx 0,4406$$

3. $k=2$ (Last point):

$$b_{2,2}=1$$

$a_{2,2} = C \;\; b_{2,2} \approx 0,5319 \;\; 1 \approx 0,5319$

An approximate solution:

$$F_{approx} \approx a_{0,2} f(t_0) + a_{1,2} f(t_1) + a_{2,2} f(t_2)$$

$$f(t_0) = 0^2 = 0, \; f(t_1) = 0.5^2 = 0.25, \; f(t_2) = 1^2 = 1$$

$$F_{approx} \approx (0,1558 \;\; 0) + (0,4406 \;\; 0,25) + (0,5319 \;\; 1)$$

$$F_{approx} \approx 0 + 0,11015 + 0,5319 \approx 0,64205$$

Find the error

Absolute error $|E|$ difference between exact and approximate solutions:

$$Error = | F_{exact} - F_{approx} |$$

$$Error \approx | 0,60179 - 0,64205 | \approx 0,04026$$

Results and discussion

In this study, both exact and approximate solutions of the Riemann–Liouville fractional integral were obtained for a selected test function. The analytical solution was derived using known closed-form expressions involving the Gamma function, which provided a reliable reference for evaluating the accuracy of numerical methods.

Numerical approximations were computed using polynomial interpolation–based quadrature formulas, including the fractional left rectangular, right rectangular, and fractional trapezoidal (L1) methods. The results show that the rectangular formulas provide a simple and computationally efficient approximation; however, their accuracy is limited due to their first-order convergence. This behavior is consistent with theoretical expectations and previously reported results in the literature.

In contrast, the fractional trapezoidal (L1) method demonstrates significantly improved accuracy. The numerical results reveal that the L1 method yields smaller absolute errors compared to the rectangular formulas for the same step size. This improvement is attributed to the use of linear interpolation, which better captures the behavior of the integrand over each subinterval while still accounting for the singular nature of the kernel.

The error analysis confirms that reducing the step size leads to a consistent decrease in the approximation error for all methods. Nevertheless, the rate of convergence is higher for the fractional trapezoidal method, making it more suitable for problems where higher accuracy is required. At the same time, the computational complexity remains moderate, which is an important advantage for large-scale numerical simulations.

Overall, the results demonstrate that polynomial interpolation–based quadrature formulas are effective tools for approximating Riemann–Liouville fractional integrals. The fractional trapezoidal (L1) method, in particular, offers a favorable balance between accuracy and computational efficiency. These properties make it a promising approach for solving practical problems in fractional differential equations and related applications.

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