

SPECTRAL EXPANSIONS FOR SECOND-ORDER ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract: This paper investigates the spectral expansion theory for second-order elliptic differential operators. The existence of eigenvalues and eigenfunctions in bounded domains is established, and solutions are expressed in terms of an orthonormal basis. The obtained results are applied to analyze boundary value problems, evaluate convergence properties, and provide a theoretical foundation for numerical methods. The influence of model parameters, regularity, and practical examples are briefly discussed. The results pave the way for future research opportunities.

Keywords: Elliptic differential operators; spectral expansions; eigenvalues; eigenfunctions; boundary value problems; self-adjoint operators; Hilbert space; spectral analysis.

Introduction This paper focuses on the spectral expansion problem for second-order elliptic differential operators. Elliptic operators play a crucial role in mathematical physics, boundary value problems, and functional analysis. Representing solutions via eigenvalues and eigenfunctions in series form holds significant theoretical and practical importance.

In recent years, spectral theory has been extended to operators with variable coefficients and complex geometric domains, allowing for a deeper analysis of solution regularity and convergence properties. Spectral expansions provide explicit representations for elliptic equations and establish a theoretical foundation for numerical algorithms.

In this work, self-adjoint operators defined in bounded domains are considered, and their discrete spectrum is analyzed. The corresponding eigenfunctions form an orthonormal basis in a Hilbert space. The theoretical part of the paper relies on references [1] and [2]. The obtained results serve as a methodological foundation for future scientific investigations.

Moreover, these findings have practical applications in educational contexts, computational modeling, and the solution of complex boundary value problems. This approach is particularly beneficial and effective for young researchers.

1-Theorem (Spectral Expansion Theorem). Let L be a second-order elliptic differential operator defined in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$Lu = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + c(x)u$$

If L is self-adjoint and satisfies the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2, \quad \lambda > 0,$$

then it possesses a discrete spectrum. That is, the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ form an increasing sequence:

$$0 < \lambda_1 < \lambda_2 < \dots \rightarrow \infty,$$

and the corresponding eigenfunctions $\{\varphi_k\}_{k=1}^{\infty}$ form an orthonormal basis in $L^2(\Omega)$. For any $f \in L^2(\Omega)$, the solution can be expressed via the spectral series:

$$u(x) = \sum_{k=1}^{\infty} \frac{(f, \varphi_k)}{\lambda_k} \varphi_k(x),$$

where (f, φ_k) denotes the Hilbert space inner product:

$$(f, \varphi_k) = \int_{\Omega} f(x) \varphi_k(x) dx .$$

Proof Outline: Self-Adjointness of the Operator. The operator L is self-adjoint in the Hilbert space $L^2(\Omega)$ if

$$(Lu, v) = (u, Lv), \quad \forall u, v \in H_0^1(\Omega).$$

This property ensures that all eigenvalues of L are real and that the spectral decomposition of the operator is valid. Self-adjointness also allows for decomposition of solutions into orthogonal eigenfunctions, forming the basis of the spectral expansion principle.

Ellipticity and Positive Definiteness: The ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j - \lambda |\xi|^2$$

implies that L is positive definite, guaranteeing that the smallest eigenvalue satisfies $\lambda_1 > 0$. Positive definiteness ensures:

1. Convergence of the spectral series,
2. Stability of the solutions,
3. Increasing order of eigenvalues, which improves solution regularity.

Compactness and Hilbert Space Argument: Since L^{-1} is compact in the Hilbert space, the Riesz–Schauder theorem implies:

1. Existence of discrete eigenvalues,
2. Corresponding eigenfunctions forming an orthonormal basis. guarantees convergence of the spectral series in $L^2(\Omega)$ and compliance with regularity conditions in $H^2(\Omega)$.

Solution via Spectral Series: For any $f \in L^2(\Omega)$, the solution is:

$$u = \sum_{k=1}^{\infty} \frac{(f, \varphi_k)}{\lambda_k} \varphi_k .$$

The series:

1. Converges in $L^2(\Omega)$,
2. Preserves regularity in $H^2(\Omega)$,
3. Controls derivatives such as gradient and Laplacian, essential for numerical computations.

Example: For $\Omega = (0,1)$ and $L = -\frac{d^2}{dx^2}$, eigenvalues are $\lambda_k = k^2 \pi^2$ and eigenfunctions $\varphi_k(x) = \sqrt{2} \sin(k\pi x)$. Any $f \in L^2(0,1)$ can be expanded in the spectral series.

Regularity and Stability: If $a_{ij} \in C(\Omega)$, the spectral series converges in $C(\Omega)$, providing smooth solutions. This:

1. Preserves operator properties,
2. Ensures solution stability,
3. Improves computational accuracy for numerical methods.

Multi-Dimensional and Geometric Extensions: Spectral properties extend to:

1. Multi-dimensional domains,
2. Complex boundaries,
3. Riemannian manifolds.

In such cases, eigenvalues and eigenfunctions reflect geometric characteristics. The spectral theorem applies, subject to smoothness and compactness requirements.

Generalization to Fractional and Nonlocal Operators: Spectral expansions are generalized to fractional elliptic and nonlocal operators:

$$L^\alpha u = (-\Delta)^\alpha u, \quad 0 < \alpha < 1,$$

where eigenvalues and eigenfunctions are defined via spectral series, although integration and regularity analysis become more complex. Applications include:

1. Anomalous diffusion,
2. Financial modeling,
3. Biological dispersal problems.

Practical Applications: Spectral expansions are applied in

1. Physical modeling,
2. Engineering computations,
3. Composite materials,
4. Heat conduction,
5. Acoustic systems.

Numerical methods, such as Galerkin and finite element methods, utilize spectral bases to improve solution accuracy.

Conclusion By analyzing the spectral structure of second-order elliptic operators, solutions can be expressed via eigenfunctions, and their regularity and stability can be assessed. The results provide a solid foundation for:

1. Theoretical investigations,
2. Numerical methods,
3. Practical modeling applications.

This approach offers a robust methodological framework for future research, education, and applied computational problems.

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