RIGOROUS DERIVATION OF A SYSTEM OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS FOR VIBRATIONS OF CURVED RODS

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Abstract

In this article, the problem of deriving a system of linear integro-differential equations describing small oscillations of curvilinear rods is considered. With a curvature along the axis of the rod, the viscoelastic properties of the material are described by the Boltzmann-Volterra hereditary model. Based on kinematic perceptions and stress-strain relationships, the equations of motion are derived using the Hamilton principle or the virtual work method. The obtained results show that in special cases (straight rod, pure elastic material) they are reduced to classical equations.

Keywords: curvilinear rod, vibration, viscoelasticity, hereditary nucleus, integro-differential equations, Hamilton's principle.

Introduction

Curved rods (such as arch-shaped elements, ring segments, skew frames, etc.) are widely encountered in engineering structures. In such elements, geometric curvature significantly affects the distribution of inertial forces and internal forces, thereby altering the vibration spectrum. At the same time, in materials such as polymers, composites, and rubber–metal systems, viscoelastic effects provide energy dissipation and can substantially change the dynamic response. Therefore, developing governing equations for the vibrations of curved rods based on a hereditary viscoelastic model is a relevant and important problem.

The objective of this paper is to rigorously derive a system of linear integro-differential equations for small vibrations of a curved rod while accounting for the viscoelastic behavior of the material.

The centerline of the rod is parameterized by the arc-length coordinate $s : 0 \quad s \quad L$. The curvature of the rod's axis is $k(s)$, and its torsion (if considered) may be $\tau(s)$. The cross-sectional area is A, the second moments of area are I_1, I_2, the polar moment of inertia is J, and the density is ρ.

For small vibrations, the following generalized displacements are introduced (for example, for planar vibrations):

u=u(s,t) - axial (longitudinal) displacement along the rod axis;

w=w(s,t) - displacement in the normal (radial) direction;

$\varphi = \varphi(s,t)$ - rotation of the cross-section.

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If the model is of Timoshenko type, the shear deformation of the cross-section is also taken into account; in the Euler–Bernoulli case, $\varphi \approx w_s$ (small angles) is assumed.

Kinematic relations (linearized)

Under small-deformation assumptions, the axial strain is given by $\varepsilon(s,t) = u_s(s,t) + \kappa(s)w(s,t)$,

where $( )_s = \dfrac{\partial ( )}{\partial s}$, and $( )_t = \dfrac{\partial ( )}{\partial t}$

Bending (curvature)

$$\chi(s,t) = w_{ss}(s,t) + \kappa(s)u_s(s,t),$$

in practice, depending on the chosen geometric scheme, the expression for χ can be written more accurately (a full form may be given in curvilinear coordinates).

If the Timoshenko assumption is adopted, the shear angle is:

$$\gamma(s,t) = w_s(s,t) - \varphi(s,t)$$

Viscoelastic constitutive relation (hereditary model)
Let the material be viscoelastic, and the Boltzmann-Volterra model be adopted:

$$\sigma(t) = E_0\varepsilon(t) + \int_0^t G_E(t-\tau)\varepsilon(\tau)d\tau$$

where E_0 is the elastic modulus, and $G_E(\ )$ – is the relaxation kernel.

In rod theory, the internal forces are:

Viscoelastic constitutive relation (hereditary model)

Let the material be viscoelastic and described by the Boltzmann-Volterra model: $\sigma(t) = E_0\varepsilon(t) + \int_0^t G_E(t-\tau)\varepsilon(\tau)d\tau$

where $E_0 -$ is the elastic modulus and $G_E(\ )$ – is the relaxation kernel.

In rod theory, the internal forces are:

Axial force: $N(t) = \int_A \sigma dA,$

Bending moment: $M(t) = \int_A \sigma y dA$ (depends on the cross-sectional axes).

In the linear case, this yields:

$$N(s,t) = EA\left[\varepsilon(s,t) + \int_0^t g_N(t-\tau)\varepsilon(s,\tau)d\tau\right]$$

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$$M(s,t) = EI\left[\chi(s,t) + \int_0^t g_M(t-\tau)\chi(s,\tau)d\tau\right]$$

where g_N and g_M are kernel functions (in many cases the same kernel is adopted, i.e., $g_N = g_M = g$).

If shear deformation is taken into account:

$$Q(s,t) = kGA\left[\gamma(s,t) + \int_0^t g_Q(t-\tau)\gamma(s,\tau)d\tau\right]$$

Energies and variational basis

Kinetic energy:

$$T = \frac{1}{2}\int_0^L \rho A\left(u_t^2 + w_t^2\right)ds + \frac{1}{2}\int_0^L \rho I_r \varphi_t^2 ds$$

If only N and M are considered (axial and bending), the "elastic part" of the potential energy is:

$$\Pi_e = \frac{1}{2}\int_0^L \left(N\varepsilon + M\chi\right)ds$$

If the shear force Q (Timoshenko) is also included:

$$\Pi_e = \frac{1}{2}\int_0^L \left(N\varepsilon + M\chi + Q\gamma\right)ds$$

Here:

$$N = EA\left(\varepsilon + G[\varepsilon]\right), \quad M = EI\left(\chi + G[\chi]\right), \quad Q = kGA\left(\gamma + G[\gamma]\right)$$

Hereditary (viscoelastic) effects in variational formulations usually appear through convolution terms; in practical computations they are introduced in the form of an "operator":

$$G[\varepsilon](t) = \int_0^t g(t-\tau)\varepsilon(\tau)d\tau$$

Hamilton's principle:

$$\delta \int_{t_1}^{t_2}(T - \Pi)dt + \delta W = 0$$

where δW – is the virtual work of external forces.

Derivation of the equations of motion

Let the distributed external loads be $p_u(s,t)$ (axial) and $p_w(s,t)$ (normal). Then the equilibrium (dynamic) equations follow from the classical rod balance relations:

1Axial motion

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$$\rho A u_{tt} - \frac{\partial N}{\partial s} + \kappa(s) Q = p_u(s,t)$$

where $Q -$ is the shear force (in the Euler-Bernoulli theory, $Q -$ can be expressed in terms of M).

2. Normal (bending) motion

$$\rho A w_{tt} - \frac{\partial Q}{\partial s} - \kappa(s) N = p_w(s,t)$$

3. Moment balance (if φ is an independent variable)

$$\rho I_r \varphi_{tt} - \frac{\partial M}{\partial s} + Q = 0$$

Now, if $M, N \ and \ Q$ are expressed via hereditary convolution operators, a system of linear integro-differential equations is obtained. For example, if $g_N = g_M = g_Q = g$ is assumed:

$$N = EA\big(\varepsilon + G[\varepsilon]\big), \ \ M = EI\big(\chi + G[\chi]\big), \ \ Q = kGA\big(\gamma + G[\gamma]\big)$$

Therefore, for example, the axial equation takes the form:

$$\rho A u_{tt} - \frac{\partial}{\partial s}\big(EA(\varepsilon + G[\varepsilon])\big) + \kappa(s)kGA(\gamma + G[\gamma]) = p_u(s,t),$$

where $\varepsilon = u_s + kw, \ \ \gamma = w_s - \varphi.$

Similarly, equations for w and φ can be obtained. As a result, we arrive at:

$$\rho A u_{tt} - \frac{\partial}{\partial s}\big(EA(\varepsilon + G[\varepsilon])\big) + \kappa kGA(\gamma + G[\gamma]) = p_u,$$

$$\rho A w_{tt} - \frac{\partial}{\partial s}\big(kGA(\gamma + G[\gamma])\big) - \kappa EA(\varepsilon + G[\varepsilon]) = p_w,$$

$$\rho I_r \varphi_{tt} - \frac{\partial}{\partial s}\big(EI(\chi + G[\chi])\big) + kGA(\gamma + G[\gamma]) = 0$$

where $\chi -$ is defined according to the chosen kinematic scheme (full curved-rod theory).

Boundary and initial conditions

For the vibration problem, the initial conditions are:

$$u(s,0) = u_0(s), \qquad u_t(s,0) = v_0(s)$$

$$w(s,0) = w_0(s), \qquad w_t(s,0) = \dot{w}_0(s)$$

$$\varphi(s,0) = \varphi_0(s), \qquad \varphi_t(s,0) = \dot{\varphi}_0(s)$$

The boundary conditions depend on how the rod ends are constrained:

clamped: $u = 0,$ $w = 0,$ $\varphi = 0$

hinged: $w = 0,$ $M = 0$ (and an additional axial condition, e.g., N or uuu);

free: $N = 0,$ $Q = 0,$ $M = 0$

In the viscoelastic case, since $N, Q \text{ and } M$ and M are given by convolution operators, the same operators also appear in the boundary conditions.

Special cases and verification

If $g(t) = 0$ (i.e., there is no viscoelasticity), then

$N = EA_\varepsilon$, $M = EI_\chi$, $Q = kGA_\gamma$, and we obtain the classical equations of an elastic curved rod.

If $k(s) = 0$ (a straight rod), the system reduces to the Timoshenko/Euler–Bernoulli dynamics of a straight rod.

If shear deformation is neglected (Euler-Bernoulli assumption), $\varphi \approx w_s$ and Q can be expressed algebraically in terms of M; consequently, the system is simplified for the variables u and w.

Conclusion

In this paper, a system of linear integro-differential equations describing vibrations of curved rods based on a hereditary viscoelastic model has been derived. The resulting mathematical model is determined by the curvature of the rod geometry, the adopted kinematic assumptions (Euler–Bernoulli or Timoshenko theory), and the relaxation kernel of the material. The validity of the model is justified by showing that, in special cases (zero curvature or neglect of hereditary effects), it reduces to the classical equations of elastic dynamics. The obtained results provide a theoretical basis for analyzing vibrations of curved frames, ring-like elements, and polymer–composite structures, as well as for evaluating dynamic parameters while accounting for dissipation effects.

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