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EXISTENCE AND UNIQUENESS THEOREMS FOR THE CAUCHY PROBLEM OF FRACTIONAL DERIVATIVE DIFFUSION EQUATIONS

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Abstract: This study investigates the existence and uniqueness theorems for the Cauchy problem of fractional derivative diffusion equations. Fractional diffusion equations, involving derivatives of non-integer order, provide a generalized framework for modeling anomalous diffusion and memory effects in complex systems. The Cauchy problem is reformulated as an equivalent integral equation using Caputo or Riemann–Liouville derivatives, enabling the application of functional analytic methods such as the Banach fixed-point theorem and semigroup theory. The results guarantee the well-posedness of the problem, ensuring that solutions exist, are unique, and depend continuously on initial data. These findings are fundamental for both analytical and numerical investigations of anomalous diffusion phenomena in physics, engineering, and applied mathematics.

Keywords: Fractional diffusion equations, Cauchy problem, Caputo derivative, Riemann–Liouville derivative, existence theorem, uniqueness theorem, anomalous diffusion, memory effects, integral equation, well-posedness.

ТЕОРЕМЫ СУЩЕСТВОВАНИЯ И ЕДИНСТВЕННОСТИ ДЛЯ ЗАДАЧИ КОШИ УРАВНЕНИЙ ДИФФУЗИИ С ДРОБНЫМИ ПРОИЗВОДНЫМИ

Аннотация: В данной работе рассматриваются теоремы существования и единственности для задачи Коши уравнений диффузии с дробными производными. Уравнения дробной диффузии, включающие производные нецелого порядка, предоставляют обобщённую модель для описания аномальной диффузии и эффектов памяти в сложных системах. Задача Коши приводится к эквивалентному интегральному уравнению с использованием производных Капуто или Римана–Лиувилля, что позволяет применять функционально-аналитические методы, такие как теорема о неподвижной точке Банаха и теория полугрупп. Полученные результаты обеспечивают корректность постановки задачи, гарантируя существование и единственность решения, а также его непрерывную зависимость от начальных данных. Эти результаты имеют фундаментальное значение для аналитического и численного изучения аномальной диффузии в физике, инженерии и прикладной математике.

Ключевые слова: Уравнения дробной диффузии, задача Коши, производная Капуто, производная Римана–Лиувилля, теорема существования, теорема единственности, аномальная диффузия, эффекты памяти, интегральное уравнение, корректность постановки.

Fractional derivative diffusion equations represent a generalization of classical diffusion equations where derivatives of non-integer order are considered, allowing the modeling of anomalous diffusion and memory effects in complex media. Unlike classical diffusion, which assumes Gaussian distribution of particle displacements, fractional diffusion equations can capture subdiffusive or superdiffusive behaviors observed in various physical, chemical, and biological systems. The Cauchy problem for such equations, which consists of determining a solution from specified initial conditions, plays a central role in both theoretical analysis and applications. The

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well-posedness of this problem, including the existence and uniqueness of solutions, is fundamental for validating mathematical models and ensuring meaningful physical interpretations.

The study of fractional derivative diffusion equations typically employs Caputo or Riemann–Liouville derivatives. The Caputo derivative is particularly convenient for problems with physically interpretable initial conditions because it allows the use of integer-order initial values. Consider the fractional diffusion equation in the Caputo sense of order α∈(0,1)\alpha \in (0,1)α∈(0,1):

∂αu(x,t)∂tα=D∂2u(x,t)∂x2+f(x,t),x∈R, t>0,\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad x \in \mathbb{R}, \ t > 0,∂tα∂αu(x,t) =D∂x2∂2u(x,t) +f(x,t),x∈R, t>0,

subject to the initial condition:

u(x,0)=u0(x),x∈R,u(x,0) = u_0(x), \quad x \in \mathbb{R},u(x,0)=u0 (x),x∈R,

where u(x,t)u(x,t)u(x,t) denotes the unknown function, DDD is the diffusion coefficient, and f(x,t)f(x,t)f(x,t) is a source term. The Cauchy problem in this context aims to determine u(x,t)u(x,t)u(x,t) for all xxx and ttt based on u0(x)u_0(x)u0 (x) and f(x,t)f(x,t)f(x,t).

Existence and uniqueness theorems for such fractional problems often rely on functional analysis techniques, such as Banach fixed-point theorem, semigroup theory, and fractional integral operators. For instance, by reformulating the fractional derivative equation as an equivalent Volterra integral equation of the second kind, one can apply contraction mapping principles to establish the existence of a unique solution in an appropriate function space. Letting JαJ^\alphaJα denote the Riemann–Liouville fractional integral operator of order α\alphaα, the equation can be rewritten as:

u(x,t)=u0(x)+1Γ(α)∫0t(t−s)α−1[D∂2u(x,s)∂x2+f(x,s)]ds.u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[D \frac{\partial^2 u(x,s)}{\partial x^2} + f(x,s) \right] ds.u(x,t)=u0 (x)+Γ(α)1 ∫0t (t−s)α−1[D∂x2∂2u(x,s) +f(x,s)]ds.

This integral form is particularly advantageous because it transforms the fractional differential equation into an operator equation for which fixed-point theorems are directly applicable. Under conditions of boundedness and Lipschitz continuity of the source term f(x,t)f(x,t)f(x,t) and regularity of u0(x)u_0(x)u0 (x), one can rigorously prove that a unique solution exists in the Banach space C([0,T];L2(R))C([0,T]; L^2(\mathbb{R}))C([0,T];L2(R)) for a finite time interval [0,T][0,T][0,T].

The uniqueness of solutions ensures the stability and predictability of the model, which is critical in applications ranging from heat conduction in heterogeneous materials to pollutant transport in porous media. In addition, the maximum principle and energy estimates can be extended to fractional diffusion equations, providing further insights into solution behavior and constraints. Moreover, numerical methods, such as finite difference schemes with Grünwald–Letnikov approximations or spectral methods, rely on the underlying existence and uniqueness results to guarantee convergence and stability.

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The field has also developed generalizations to multi-dimensional domains, variable-order derivatives, and systems with coupled fractional equations. For multi-dimensional problems, the Cauchy problem can be formulated as:

∂αu(x,t)∂tα=∇⋅(D(x)∇u(x,t))+f(x,t),x∈Rn, t>0,\frac{\partial^\alpha u(\mathbf{x},t)}{\partial t^\alpha} = \nabla \cdot (D(\mathbf{x}) \nabla u(\mathbf{x},t)) + f(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0,∂tα∂αu(x,t) =∇⋅(D(x)∇u(x,t))+f(x,t),x∈Rn, t>0,

with initial condition u(x,0)=u0(x)u(\mathbf{x},0) = u_0(\mathbf{x})u(x,0)=u0 (x). The existence and uniqueness theorems are then established using Sobolev spaces and fractional semigroup theory, often requiring stricter regularity conditions on u0(x)u_0(\mathbf{x})u0 (x) and D(x)D(\mathbf{x})D(x).

Applications of these theorems extend beyond theoretical analysis. For example, in viscoelastic materials, the stress-strain relationship can be modeled using fractional diffusion equations, and the uniqueness theorem ensures that the response is deterministically defined for given initial stress conditions. Similarly, in hydrology and finance, anomalous diffusion models depend on the rigorous solvability of fractional Cauchy problems for accurate predictions.

The rigorous establishment of existence and uniqueness for the Cauchy problem of fractional derivative diffusion equations requires precise functional analytic frameworks. One commonly used approach involves casting the problem in an appropriate Banach or Hilbert space setting. Let us consider the fractional diffusion equation with Caputo derivative of order 0<α<10 < \alpha < 10<α<1:

∂tαu(x,t)=D∂xxu(x,t)+f(x,t),x∈R, t>0,\partial_t^\alpha u(x,t) = D \partial_{xx} u(x,t) + f(x,t), \quad x \in \mathbb{R},\ t>0, ∂tα u(x,t)=D∂xx u(x,t)+f(x,t),x∈R, t>0,

subject to the initial condition

u(x,0)=u0(x),x∈R.u(x,0) = u_0(x), \quad x \in \mathbb{R}.u(x,0)=u0 (x),x∈R.

By applying the fractional integral operator JαJ^\alphaJα and the properties of Caputo derivatives, the equation can be reformulated into an equivalent Volterra integral equation:

u(x,t)=u0(x)+1Γ(α)∫0t(t−s)α−1[D∂xxu(x,s)+f(x,s)]ds.u(x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[D \partial_{xx} u(x,s) + f(x,s) \right] ds.u(x,t)=u0 (x)+Γ(α)1 ∫0t (t−s)α−1[D∂xx u(x,s)+f(x,s)]ds.

This transformation is pivotal because the integral form allows the application of fixed-point theorems. In particular, the Banach fixed-point theorem ensures the existence of a unique solution under certain Lipschitz continuity conditions on f(x,t)f(x,t)f(x,t) with respect to uuu. Specifically, if there exists a constant L>0L>0L>0 such that

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∥f(x,t,u1)−f(x,t,u2)∥≤L∥u1−u2∥,∀u1,u2∈X,\| f(x,t,u_1) - f(x,t,u_2) \| \leq L \| u_1 - u_2 \|, \quad \forall u_1, u_2 \in X,∥f(x,t,u1)−f(x,t,u2)∥≤L∥u1 −u2 ∥,∀u1 ,u2 ∈X,

for a suitable function space XXX, then a unique solution u∈C([0,T];X)u \in C([0,T]; X)u∈C([0,T];X) exists. The contraction mapping argument hinges upon defining an operator

T[u](x,t)=u0(x)+1Γ(α)∫0t(t−s)α−1[D∂xxu(x,s)+f(x,s,u(x,s))]ds\mathcal{T}[u](x,t) = u_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[D \partial_{xx} u(x,s) + f(x,s,u(x,s)) \right] dsT[u](x,t)=u0 (x)+Γ(α)1 ∫0t (t−s)α−1[D∂xx u(x,s)+f(x,s,u(x,s))]ds

and proving that T\mathcal{T}T is a contraction on a closed, bounded subset of C([0,T];X)C([0,T]; X)C([0,T];X).

Another approach involves spectral decomposition. Assuming that u0(x)u_0(x)u0 (x) and f(x,t)f(x,t)f(x,t) are sufficiently regular, the Laplace transform in time can be applied, yielding:

sαu~(x,s)−sα−1u0(x)=D∂xxu~(x,s)+f~(x,s),s^\alpha \tilde{u}(x,s) - s^{\alpha-1} u_0(x) = D \partial_{xx} \tilde{u}(x,s) + \tilde{f}(x,s),sαu~(x,s)−sα−1u0 (x)=D∂xx u~(x,s)+f~ (x,s),

where u~(x,s)\tilde{u}(x,s)u~(x,s) and f~(x,s)\tilde{f}(x,s)f~ (x,s) denote Laplace transforms with respect to ttt. The transformed equation reduces to a classical elliptic problem in xxx with parameter sαs^\alphasα, allowing for solution representation via Green's functions or Fourier transforms. Inverting the Laplace transform then provides the solution u(x,t)u(x,t)u(x,t) in the time domain.

Explicit solution representations for specific cases are also instructive. For instance, for the homogeneous equation (f≡0f \equiv 0f≡0) with initial condition u0(x)=δ(x)u_0(x) = \delta(x)u0 (x)=δ(x), the solution is given by the fundamental solution of the fractional diffusion equation:

u(x,t)=12t−α/2M(∣x∣tα/2;α),u(x,t) = \frac{1}{2} t^{-\alpha/2} M \left(\frac{|x|}{t^{\alpha/2}}; \alpha \right),u(x,t)=21 t−α/2M(tα/2∣x∣ ;α),

where M(⋅;α)M(\cdot;\alpha)M(⋅;α) is the M-Wright function, which generalizes the Gaussian kernel of classical diffusion. This example highlights how fractional derivatives introduce heavy-tailed, non-Gaussian spreading of the diffusive quantity, capturing subdiffusive (α<1\alpha < 1α<1) dynamics.

The uniqueness of the solution ensures stability: any two solutions with the same initial condition must coincide. Formally, if u1u_1u1 and u2u_2u2 satisfy the fractional diffusion equation with identical initial conditions, then the difference w=u1−u2w = u_1 - u_2w=u1 −u2 satisfies

∂tαw(x,t)=D∂xxw(x,t),w(x,0)=0.\partial_t^\alpha w(x,t) = D \partial_{xx} w(x,t), \quad w(x,0) = 0.∂tα w(x,t)=D∂xx w(x,t),w(x,0)=0.

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Applying the Laplace transform or energy estimates leads to w(x,t)≡0w(x,t) \equiv 0w(x,t)≡0, confirming uniqueness.

The theoretical framework extends naturally to multi-dimensional fractional diffusion equations:

∂tαu(x,t)=∇⋅(D(x)∇u(x,t))+f(x,t),x∈Rn,t>0,\partial_t^\alpha u(\mathbf{x},t) = \nabla \cdot (D(\mathbf{x}) \nabla u(\mathbf{x},t)) + f(\mathbf{x},t), \quad \mathbf{x} \in \mathbb{R}^n, t > 0,∂tα u(x,t)=∇⋅(D(x)∇u(x,t))+f(x,t),x∈Rn,t>0,

with initial data u(x,0)=u0(x)u(\mathbf{x},0) = u_0(\mathbf{x})u(x,0)=u0 (x). Existence and uniqueness are typically established in Sobolev spaces Hk(Rn)H^k(\mathbb{R}^n)Hk(Rn), exploiting properties of elliptic operators and fractional semigroups. Moreover, variable-order derivatives α=α(t)\alpha = \alpha(t)α=α(t) introduce time-dependent memory effects, and corresponding existence theorems rely on generalized Gronwall inequalities adapted to fractional calculus.

In applied contexts, the practical significance of these theorems is evident. Numerical simulations using Grünwald–Letnikov approximations, finite element methods, or spectral schemes assume a unique solution exists; convergence analyses are valid only under these theoretical guarantees. Applications in viscoelastic materials, porous media flow, and anomalous transport phenomena all benefit from a rigorous understanding of solution existence and uniqueness.

Conclusion

The study of existence and uniqueness theorems for the Cauchy problem of fractional derivative diffusion equations demonstrates the rigorous mathematical foundation underlying anomalous diffusion processes. Fractional derivatives, particularly in the Caputo or Riemann–Liouville sense, enable modeling of memory effects and subdiffusive or superdiffusive phenomena that classical diffusion equations cannot capture. By reformulating the problem as an equivalent integral equation and employing functional analytic tools such as Banach fixed-point theorem and semigroup theory, one can establish the well-posedness of the problem. Uniqueness ensures stability and predictability, while existence guarantees that the modeled physical or engineering systems behave consistently under specified initial conditions. These results form the basis for both analytical solutions and reliable numerical simulations in various applied fields, including viscoelastic materials, porous media, and anomalous transport phenomena.

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