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FINITE ELEMENT METHOD FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

Doniyorov N. N.
Asia International University,
74, Gijduvon str., Bukhara 200114, Uzbekistan
email: doniyorovnn@mail.ru

Abstract

In this work, an integral equation is presented for solving second-order linear differential equations of elliptic type using the finite element method. In addition, basis functions of two variables are constructed in a bounded domain on the plane.

Key words. approximation, basis functions, finite elements, PDE, mixed boundary value problem.

We consider a linear partial differential equation of the second order

$$-\overset{2}{\underset{i=1}{}}\frac{\partial}{\partial x_i}\ c\frac{\partial u}{\partial x_i}\ +\overset{2}{\underset{i=1}{}}b_i\frac{\partial u}{\partial x_i}+au=f \tag{1}$$

in a domain $\Omega=[0,1]\ \ [0,1]$ bounded on a plane. Let the classical solution $u(x_1,x_2)$ satisfying the mixed boundary conditions

$$u=u_D,\ x\ \ \Gamma_0, \tag{2}$$

$$c\ \ u\ \ \nu+\sigma u=\mu,\ x\ \ \Gamma_1 \tag{3}$$

exist and be unique to equation (1). Here, c,a,f are scalar functions, $b=(b_1,b_2)$ is a vector function, a sufficiently smooth function of variable $x=(x_1,x_2)\ \ \overline{\Omega}$, is a gradient differential operator, $c(x)>0\left(x\ \ \overline{\Omega}\right)$, $\Gamma=\partial\Omega=\Gamma_0\ \ \Gamma_1$, $\Gamma_0\ \ \Gamma_1=\ \ $, u_D,σ,μ are given continuous functions, $\nu=\nu(x)$ is the external unit normal vector at $x\ \ \Gamma_1$.

The following theorem holds.

Theorem 1. Constructing the approximate solution u_h of the mixed boundary value problem $(1)-(3)$ using the finite element method leads to finding such a function $u_h\ \ V_h$ in which the equality

$$\underset{\Omega_h}{}\left(c\ \ u_h\ \ \vartheta_h+b\ \ u_h\vartheta_h+au_h\vartheta_h\right)dx+\underset{\Gamma_1^h}{}\sigma u_h\vartheta_h dx$$

$$=\underset{\Omega_h}{}f\vartheta_h dx+\underset{\Gamma_1^h}{}\mu\vartheta_h dx$$

is valid for an arbitrary $\vartheta_h\ \ V_h^0$ (see: [1]). Where, $\Omega_h=\bigcup_{\tau\ \ \tau_h}\tau$, τ_h is the partition of the domain Ω into Lagrangian finite elements of the same type $\tau,\Gamma_0^h,\Gamma_1^h$ is the parts of the boundary of the domain Ω_h corresponding to Γ_0 and Γ_1,

$$V_h=\left\{\vartheta\ \ S_h:\vartheta(x)=u_D(x),x\ \ \gamma_h\right\},$$

$$V_h^0=\left\{\vartheta\ \ S_h:\vartheta(x)=0,x\ \ \gamma_h\right\},$$

$$S_h = \left\{\vartheta \quad C\left(\overline{\Omega}_h\right) \quad H^1\left(\Omega_h\right) : \vartheta\big|_\tau \quad P_\tau, \forall\tau \quad \tau_h\right\},$$

P_τ is the space of finite-dimensional functions in τ, γ_h is the set of interpolation nodes corresponding to Γ_0^h.

It is known that the approximate solution $u_h \quad V_h$ of the mixed boundary value problem $(1)-(3)$ is sought in the form

$$u_h\left(x_1, x_2\right) = \overset{m}{\underset{i=0}{}} \; \overset{n}{\underset{j=0}{}} \alpha_{ij}\varphi_{ij}\left(x_1, x_2\right), \tag{4}$$

here α_{ij} are unknown coefficients, and $\varphi_{ij}\left(x_1, x_2\right)$ are linearly independent functions.

The functions $\varphi_{ij}\left(x_1, x_2\right)$ satisfy the following conditions:

$i)\varphi_{ij}$ functions are piecewise-linear functions in domain Ω;

$ii)\varphi_{ij} \quad C(\Omega)(i = 0,1,\ldots,m, j = 0,1,\ldots,n)$;

$iii)\varphi_{ij}\left(i = 0,1,\ldots,m, j = 0,1,\ldots,n\right)$ functions are satisfying the relation

$$\varphi_{ij}\left(x_p, x_q\right) = \begin{array}{l} 1, p = i \,\text{and}\, q = j, \\ 0, \text{otherwise}, \end{array}$$

i.e.,

$$\varphi_{ij}\left(x_p, x_q\right) = \delta_{ip}\delta_{jq}\left(p = 0,1,\ldots,m, q = 0,1,\ldots,n\right),$$

where δ_{ip} and δ_{jq} are Kronecker symbols.

In this work, we construct two-variable trigonometric basis functions to find the approximate solution (4) in analytical form. For this, we will use the results obtained in the work [2].

Theorem 2. Piecewise-linear functions φ_{ij} are defined in the domain Ω have the form:

$$\varphi_{ij}\left(x_1, x_2\right) = \begin{array}{l}
\dfrac{\sin\left(x_1 - x_{1,i-1}\right)\sin\left(x_2 - x_{2,j-1}\right)}{\sin\left(x_{1,i} - x_{1,i-1}\right)\sin\left(x_{2,j} - x_{2,j-1}\right)}, x_{1,i-1} \quad x \quad x_{1,i}, x_{2,j-1} \quad x \quad x_{2,j}, \\
\dfrac{\sin\left(x_1 - x_{1,i-1}\right)\sin\left(x_2 - x_{2,j+1}\right)}{\sin\left(x_{1,i} - x_{1,i-1}\right)\sin\left(x_{2,j} - x_{2,j+1}\right)}, x_{1,i-1} \quad x \quad x_{1,i}, x_{2,j} \quad x \quad x_{2,j+1}, \\
\dfrac{\sin\left(x_1 - x_{1,i+1}\right)\sin\left(x_2 - x_{2,j-1}\right)}{\sin\left(x_{1,i} - x_{1,i+1}\right)\sin\left(x_{2,j} - x_{2,j-1}\right)}, x_{1,i} \quad x \quad x_{1,i+1}, x_{2,j-1} \quad x \quad x_{2,j}, \\
\dfrac{\sin\left(x_1 - x_{1,i+1}\right)\sin\left(x_2 - x_{2,j+1}\right)}{\sin\left(x_{1,i} - x_{1,i+1}\right)\sin\left(x_{1,j} - x_{1,j+1}\right)}, x_{1,i} \quad x \quad x_{1,i+1}, x_{2,j} \quad x \quad x_{2,j+1}, \\
0, \text{ otherwise},
\end{array}$$

$$i = 1, 2, \ldots, m-1, \; j = 1, 2, \ldots, n-1,$$

where $\left(x_{1,i},x_{2,j}\right)\quad\Omega$.

It should be noted that the functions $\varphi_{ij}\left(x_1,x_2\right)$ form a finite dimensional linear space in the domain Ω, and therefore these functions are uniquely defined by their values at the nodes $\left(x_{1,i},x_{2,j}\right)\quad\Omega$.

To geometrically represent the basis functions $\varphi_{ij}\left(x_1,x_2\right)$, $i=1,2,\ldots,m-1,\quad j=1,2,\ldots,n-1$, we present their graphs for the case $m=n=10$ in Fig.1.

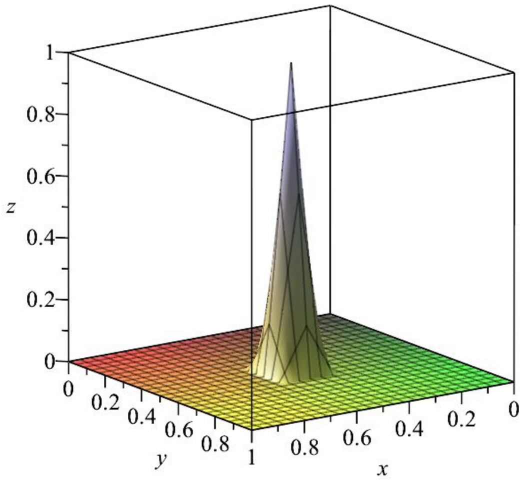

Figure 1. This figure shows the graph of the basis function $\varphi_{ij}\left(x_1,x_2\right)\left(i=1,2,\ldots,m-1,j=1,2,\ldots,n-1\right)$.

By employing a system of linearly independent functions $\varphi_{ij}\left(x_1,x_2\right)$, a highly accurate approximate solution to the mixed boundary value problem $(1)-(3)$ can be obtained using the finite element method.

References

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