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SPECTRAL PROPERTIES OF LOTKA VOLTERRA DYNAMIC SYSTEMS

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Abstract: We study the spectral properties of Jacobians at fixed points in discrete Lotka--Volterra maps associated with two types of homogeneous tournaments. We used python scipt to classify 3--support and compute their spectra when \$a_{ki}=\pm 1\$. Our results reveal general eigenvalue patterns, independent of the tournament size, with applications to stability and structural dynamics of systems.

Keywords: Lotka--Volterra map; homogeneous tournament; fixed point; Jacobian spectrum;

Introduction

By the 1920s, A. J. Lotka and V. Volterra, independently of each other, began to publish their studies in different scientific fields, respectively in autocatalytic reactions and in the evolution of biological populations, using the same differential equations. In the context of discrete Lotka--Volterra dynamics, tournaments encode interaction coefficients, allowing us to analyze the stability of equilibria and the long-term behavior of competitive systems.

Preliminaries

In its general form, a Lotka-Volterra system in dimension m is a dynamical system, described by the following system

$$V(x)=\left(x_1^{'},\dots,x_m^{'}\right),\ x_i^{'}=x_i\left(1+\sum_{j=1}^{m} a_{ij}x_j\right).$$

We consider a general skew-symmetric matrix of order m :

$$A=\left[a_{ij}\right] \text{ such that } a_{ij}=-a_{ji},\ a_{ii}=0$$

From this condition, it follows that all the diagonal elements of the matrix must be zero and the matrix takes the following form:

$$\begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1m} \\ -a_{12} & 0 & a_{23} & \dots & a_{2m} \\ -a_{13} & -a_{23} & 0 & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1m} & -a_{2m} & -a_{3m} & \dots & 0 \end{bmatrix}$$

If $a_{ki}>0$ the edge of the tournament is directed from x_i to x_k, else direction is opposite. Let x_1,x_2 be the vertices of a tournament. The notation $x_1 \to x_2$ means that the edge connecting x_1 and x_2 is directed from x_1 to x_2. A finite sequence of vertices $x_1 \to x_2 \to \dots \to x_p$ is called a path if $x_i \neq x_j$ for all $i\neq j$. A cycle is a closed path, i.e., $x_p=x_1$.

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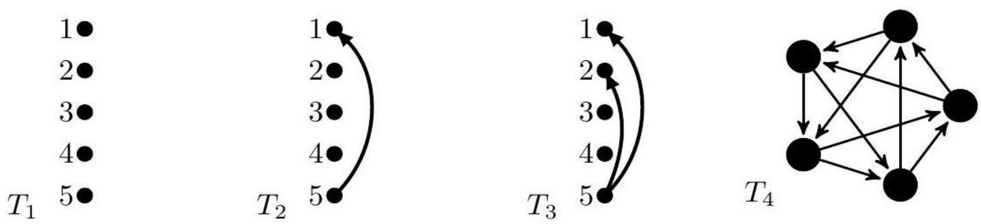
A tournament is called strong if, for any vertices $x,y \in Y$, there exists a path from x to y.

A tournament that contains no cycles is called transitive.
A tournament is called homogeneous if every sub-tournament is either strong or transitive. A fixed point x^* satisfies $V(x^*)=x^*$. If x^* has exactly k nonzero coordinates, it is called a k-support fixed point. Of special interest are the 3 -support fixed points corresponding to 3 -cycles in the tournament.

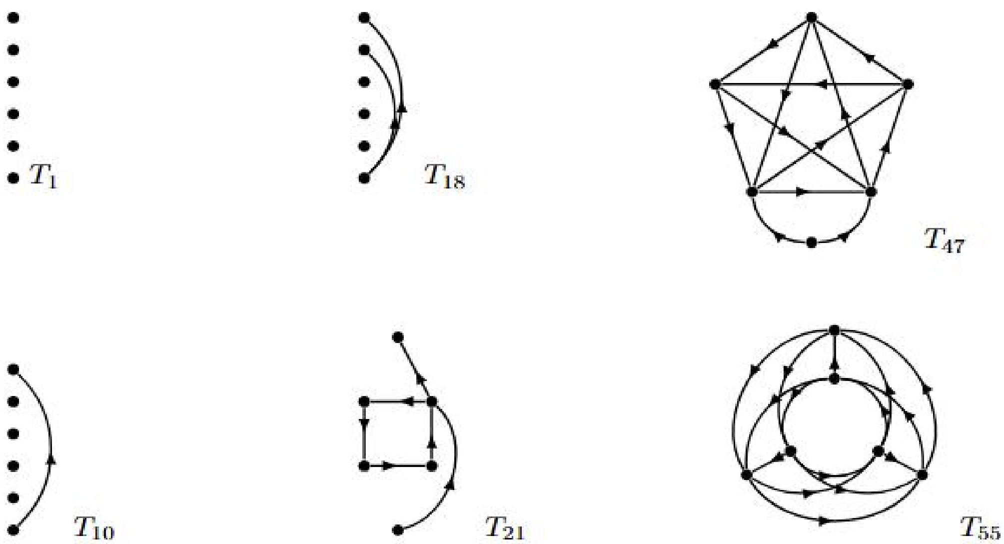
Two tournaments are isomorphic if there is a relabelling of the vertices that preserves the directed edges. That is, if there exists a bijection (one-to-one mapping) $f: V_1 \to V_2$ between the vertex sets of two tournaments T_1 and T_2 such that:

$$x \to y \text{ in } T_1 \Leftrightarrow f(x) \to f(y) \text{ in } T_2.$$

If no such mapping exists, the tournaments are non-isomorphic.
There are 12 non-isomorphic tournaments m=5 and 4 of them are homogeneous. The following drawing drawings are used to illustrate tournaments. Not all of the arcs have been included in the drawings; if an arc joining two nodes has not been drawn, then it should be understood that the arc is oriented from the higher node to the lower node.



For m=5 both homogeneous ones and non homogeneous ones are studied before. For m=6 in J.Moon 56 of them separated as non isomorphic. Then we identified 6 of them as homogeneous. We left numeration as given in J.Moon.



We tried to find general pattern for tournaments form of T_1 and T_{10} when m is arbitrary.

3 Methodology

Let

$$S^{(n-1)}=\left\{x=(x_1,\dots,x_n):\sum_{i=1}^{n} x_i=1,x_i\geq 0\right\}$$

be the simplex in $\mathbb{R}^n$.
We consider the mapping $V:S^{(n-1)}\to S^{(n-1)}$ defined by

$$x_k'=x_k\cdot\left(1+\sum_{i=1}^{n} a_{ki}x_i\right),\ k=1,\dots,n, \tag{1}$$

where $a_{ki}=-a_{ik}, |a_{ki}|\leq 1$, and $Vx=(x_1',\dots,x_n')$.
From the form of (1), if $x_k=0$ then automatically $x_k'=0$.

Let $I=\{1,\dots,n\}$ and let $a\subset I$. Denote by T_a the face of the simplex

$$S^{n-1}=\left\{x=(x_1,\dots,x_n)\in\mathbb{R}^n:\sum_{i=1}^{n} x_i=1,x_i\geq 0\right\}$$

spanned by the vertices $\{e_i\}_{i\in a}$. The restriction of V to T_a will be denoted by V_a.

Consider the matrix A_a, which is obtained from the matrix A by replacing with zero all entries a_{ki} where $(k,i)\notin a\times a$. Note that $V_a: T_a\to T_a$ can also be represented in the form (1), with the matrix A replaced by A_a.

Lemma. Let $A=(a_{ij})$ be a skew-symmetric matrix, with $|a_{ki}|<1$. Then

$$P=\{x\in S^{n-1}:Ax>0\}\neq\emptyset,\ Q=\{x\in S^{n-1}:Ax<0\}\neq\emptyset,$$

where inequalities are understood coordinatewise in $\mathbb{R}^n$.
Moreover, P and Q consist of fixed points of the mapping V.

The Jacobian matrix of the map is given by

$$J(x)=\det\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m}\end{bmatrix}.$$

By solving $\det|J(x)-\lambda I|=0$ equation, we get eigenvalues and get Jacobians spectrum.
Theorem[Discrete Hartman-Grobman / Hyperbolicity and spectral modulus] Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a C^1 map and let $x^*\in\mathbb{R}^n$ be a fixed point of $F, F(x^*)=x^*$. Let $J=DF(x^*)$ denote the Jacobian matrix at x^* and let $\sigma(J)=\{\lambda_1,\dots,\lambda_n\}$ be its spectrum.

Assume that J is hyperbolic, i.e. no eigenvalue lies on the complex unit circle:

$$\forall\lambda\in\sigma(J)\ |\lambda|\neq 1.$$

Then there exist neighborhoods U of x^* and V of 0 and a homeomorphism

$$h: U\to V,\ h(x^*)=0,$$

such that for all $x\in U$,

$$h(F(x))=Jh(x).$$

Consequently the local qualitative dynamics of F near x^* are topologically conjugate to the linear map $y\mapsto Jy$.

Moreover, the spectral moduli classify the local stability as follows:

1. If $|\lambda|<1$ for every $\lambda\in\sigma(J)$, then x^* is locally asymptotically stable (a local attractor).
2. If $|\lambda|>1$ for every $\lambda\in\sigma(J)$, then x^* is unstable and is a local repeller.
3. If some eigenvalues satisfy $|\lambda|<1$ and others $|\lambda|>1$, then x^* is a saddle (has stable and unstable manifolds).

4 Main results

Our goal is to find spectrum of Jacobian of fixed points for tournaments in form of T1 when m is arbitrary.

At first we tried to find when coefficients are ±1 to identify the pattern. First we look transitive tournaments.

For any tournament, the vertices of the simplex are always invariant. That means $e_k, k=\overline{1,m}$ are fixed points of maps of the tournaments.
Theorem 1. Let $a_{ki}=\pm1$. Then its spectrum of Jacobian is

$$\sigma\big(J(e_k)\big)=\{1,\underbrace{2,\dots,2}_{m-k},\underbrace{0,\dots,0}_{k-1}\}.$$

Proof. In this case coefficients are $a_{ki}=\pm1$. Its skew-symmetric matrix takes form below

$$A=\begin{pmatrix}0 & -1 & -1 & \dots & -1\\ 1 & 0 & -1 & \dots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & 0\end{pmatrix}$$

$$J(e_1)=\begin{pmatrix}1 & -1 & -1 & \dots & -1\\ 0 & 2 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 2\end{pmatrix}$$

By calculating eigenvalues we can get

$$\sigma\big(J(e_1)\big)=\{1,2,2,\dots,2\}.$$

With same calculations we can find

$$\sigma\big(J(e_k)\big)=\{1,\underbrace{2,\dots,2}_{m-k},\underbrace{0,\dots,0}_{k-1}\}.$$

This result holds true for any tournaments. Map of the transitive tournament does not have any inner fixed points. That means spectrum of map of transitive tournaments only above.

Conclusion

We have systematically analyzed the spectral properties of Jacobians at 3-support fixed points in discrete Lotka--Volterra maps associated with homogeneous tournaments when $a_{ki}=\pm 1$. Our results demonstrate universal eigenvalue patterns that are independent of tournament size, with applications to stability analysis and structural dynamical systems. Because of we have seen these tournaments in the simplex we have one $\lambda=1$ eigenvalue. For transitive tournaments, the spectrum at fixed points e_k consists of eigenvalues 1 and $1\pm a_{kj}$ for various j.

LITERATURE

1. Ganikhodzhaev R.N. A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems, Math. Notes, 56 (5-6), (1994) pp.1125-1131.

2. Ganikhodzhaev R.N. Quadratic stochastic operators, Lyapunov function and tournaments, Acad. Sci. Sb. Math., 76(2), p. 489-506. (1993)

3. Ganikhodzhaev R.N., Tadzhieva M.A. Stability of fixed points of discrete dynamic systems of Volterra type. AIP Conference Proceedings, 2021. V. 2365. P. 060005-1 $-$ 060005-7. https://doi.org/10.1063/5.0057979. (Scopus. IF=0.7).

4. Kh. Koh and F. Dong and E.G. Tay. Introduction to graph theory. World Scientific. p. 308. (2024)

5. Moon,J.~W. Topics on Tournaments, New York: Holt, {\it Rinehart and Winston}, 1968, p. 112.