JOURNAL OF MULTIDISCIPLINARY SCIENCES AND INNOVATIONS

VOLUME 04, ISSUE 09
MOPNTHLY JOURNALS

ISSN NUMBER: 2751-4390
IMPACT FACTOR: 9,08

MIXED PROBLEM FOR THE FRACTIONAL ORDER WAVE PROPAGATION EQUATION

Shoxsanam Abdimurodova Anvar qizi

Teacher at Karshi State University,

Berdiyev Mirjalol,Nuraliyeva E'zoza

Students at Karshi State University.

Abstract: This article investigates a mixed boundary value problem for the fractional-order wave propagation equation. Differential equations involving fractional derivatives provide a more accurate description of complex dynamic processes compared to classical wave equations. The mathematical model of the fractional-order wave equation is presented and its physical interpretation is discussed. Based on the given initial and boundary conditions, the existence and uniqueness of the solution to the mixed problem are analyzed. Analytical and numerical solution methods are compared and discussed, and several examples are provided to verify the obtained results. The findings of the study contribute to a deeper understanding of physical processes governed by fractional differential equations and offer a theoretical foundation for their practical modeling. This problem was solved using Fourier's method, one of the most common methods in mathematical physics.

Keywords: Fractional derivative and integral, mixed problem, Fourier method, fractional derivative in the sense of Caputo.

INTRODUCTION

The study of the symmetry properties of differential equations containing fractional derivatives is currently an urgent problem in connection with the increasingly widespread use of such equations as mathematical models of various processes with anomalous kinetics. In addition, unlike the classical integral derivative, there are many non-uniform definitions of fractional derivatives [1,2,3,4], which leads to a variety of fractional-order differential equations, which are close in form but significantly different in properties. The use of fractional derivatives of a function with respect to a function, in general, allows us to expand the class of possible substitutions of variables, considering them as a new type of equivalence transformations. Many mathematicians have been engaged in scientific work with fractional-order partial differential equations. This area has appeared since the beginning of the 18th century, and was introduced by J. Fourier, J. Liuwul, B. Riemann at that time. Fractional partial differential equations are widely used in physics, biology, medicine, economics and finance, computer science, and artificial intelligence.

The main part. Let us consider the problem of finding a function $u(x,t)$ in the domain D,
$D = \{(x,t) \in R^2, 0 < x < l, 0 < t < T\}$

$$D_t^{\alpha} u(x,t) - a^2 u_{xx}(x,t) = f(x),\ 0 < x < l, 0 < t < T \qquad (1)$$

JOURNAL OF MULTIDISCIPLINARY SCIENCES AND INNOVATIONS
VOLUME 04, ISSUE 09
MOPNTHLY JOURNALS
ISSN NUMBER: 2751-4390
IMPACT FACTOR: 9,08

That satisfies the equation

$$u(x,0)=\varphi(x) \quad u_t(x,0)=\psi(x) \quad 0\le x\le l \tag{2}$$

and the initial and

$$u(0,t)=0, \; u(l,t)=0 \tag{3}$$

boundary conditions. Here $\varphi(x),\psi(x),f(x)$ are given functions, a is constant, $1<\alpha<2$.

Theorem. Let us consider, $\varphi(x),\psi(x)\in C^2[0,l]$ and has a second-order piecewise-continuous derivative,and let $\varphi(0)=\varphi(l)=0, \varphi''(0)=\varphi''(l)=0$ and $\psi(0)=\psi(l)=0, \psi''(0)=\psi''(l)=0$ .Then,there are $u(x,t)$ unique solution to (1)-(3) mixed problem and it will be as follows:

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n E_\alpha\left(-\left(\frac{\pi n a}{l}\right)^2 x^\alpha\right)+\psi_n x E_{\alpha,2}\left(-\left(\frac{\pi n a}{l}\right)^2 x^\alpha\right)\sin\frac{\pi n}{l}x\right)+$$
$$+\sum_{n=1}^{\infty} f_n t^\alpha E_{\alpha,\alpha+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^\alpha\right)\sin\frac{\pi n}{l}x.$$

METHODS

Proof. To prove the theorem, we will use one of the common methods for solving partial differential equations, namely the separation of variables, or the Fourier method. We look for the solution to problems(1)-(3) in the following form,where $v(x,t)$ is the solution to problem

$$D_t^\alpha v(x,t)-a^2 v_{xx}(x,t)=0 \quad 0<t<T \tag{5}$$

$$v(x,+0)=\varphi(x) \qquad v_t(x,+0)=\psi(x) \quad 0\le x\le l \tag{6}$$

$$v(0,t)=0 \quad v(l,t)=0 \quad 0\le t\le T \tag{7}$$

And $w(x,t)$ is the solution to problem

$$D_t^\alpha w(x,t)-a^2 w_{xx}(x,t)=f(x) \tag{8}$$

$$w(x,+0)=0 \qquad w_t(x,+0)=0 \quad 0\le x\le l \tag{9}$$

$$w(0,t)=0 \quad w(l,t)=0 \quad 0\le t\le T \tag{10}$$

To solve problems (1)-(3), it is sufficient to solve the two auxiliary problems above. In this case, we solve problems (1)-(3) separately for the two cases: homogeneous and non-homogeneous.

Thus, the formal solution to problem (5)-(7) is as follows:

$$v(x,t)=\sum_{n=1}^{\infty}\varphi_n E_\alpha\left(-\left(\frac{\pi n a}{l}\right)^2 x^\alpha\right)+\psi_n x E_{\alpha,2}\left(-\left(\frac{\pi n a}{l}\right)^2 x^\alpha\right)\sin\frac{\pi n}{l}x$$

Now let's look at the non-homogeneous case. We also use the Fourier method to solve problems (8)-(10),that is, we look for the solution in the form

$$w(x,t)=\sum_{n=1}^{\infty}T_n(t)\sin\frac{\pi n}{l}x$$

and we obtain the following solution.

$$w(x,t)=\sum_{n=1}^{\infty} f_n t^{\alpha} E_{\alpha,\alpha+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^{\alpha}\right)\sin\frac{\pi n}{l}x \qquad (11)$$

Now we show that this series converges uniformly. To do this,we define the partial sum of the series (11) as

$$W_j(x,t)=\sum_{n=1}^{j} f_n t^{\rho} E_{\rho,\rho+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^{\rho}\right)\sin\frac{\pi n x}{l}$$

From the estimate of the Mittag-Lefler function

$$\left|E_{\rho,\mu}(-z)\right|\le\frac{1}{1+z},$$

The linear convergence of the series (11) follows.The smooth convergence of $w_{xx}(x,t)$ follows from the value in

$$\frac{\partial}{\partial x^2}W_j(x,t)=\sum_{n=1}^{j} f_n\left(\frac{\pi n}{l}\right)^2 t^{\rho} E_{\rho,\rho+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^{\rho}\right)\sin\frac{\pi n x}{l}$$

and the property of

$$\left(\frac{\pi n}{l}\right)^2 E_{\rho,\rho+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^{\rho}\right)\le 1$$

Moreover, it follows from equality (8) that

$$D_t^{\rho}W_j(x,t)=\frac{\partial^2}{\partial x^2}W_j(x,t)+\sum_{k=1}^{j} f_k(t)\sin\frac{\pi n x}{l},t>0$$

is

$$D_t^{\rho}w(t)\in C((0,l)\times(0,T)).$$

Therefore, from the above considerations, it follows that function (11) is a solution to problem (8) – (10). Generalizing these solutions, we obtain the following solution for problems (1) – (3):

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n E_{\alpha}\left(-\left(\frac{\pi n a}{l}\right)^2 x^{\alpha}\right)+\psi_n x E_{\alpha,2}\left(-\left(\frac{\pi n a}{l}\right)^2 x^{\alpha}\right)\sin\frac{\pi n}{l}x\right)+$$
$$+\sum_{n=1}^{\infty} f_n t^{\alpha} E_{\alpha,\alpha+1}\left(-\left(\frac{\pi n a}{l}\right)^2 t^{\alpha}\right)\sin\frac{\pi n}{l}x$$

The theorem is proved.

RESULTS.

The following results were obtained in solving this mixed problem:

Theorem. Suppose, $\varphi(x),\psi(x)\in C^2[0,l]$ and has a second-order piecewise-continuous derivative,let $\varphi(0)=\varphi(l)=0,\varphi''(0)=\varphi''(l)=0$ and $\psi(0)=\psi(l)=0,\psi''(0)=\psi''(l)=0$ be. Then the mixed problem (1)-(3) has a unique solution and looks like this:

JOURNAL OF MULTIDISCIPLINARY SCIENCES AND INNOVATIONS

VOLUME 04, ISSUE 09
MOPNTHLY JOURNALS

ISSN NUMBER: 2751-4390
IMPACT FACTOR: 9,08

$$u(x,t)=\sum_{n=1}^{\infty}\left[\varphi_n E_\alpha\left(-\left(\frac{\pi n\alpha}{l}\right)^2 x^\alpha\right)+\psi_n x E_{\alpha,2}\left(-\left(\frac{\pi n\alpha}{l}\right)^2 x^\alpha\right)\right]\sin\frac{\pi n}{l}x+$$

$$+\sum_{n=1}^{\infty} f_n t^\alpha E_{\alpha,\alpha+1}\left(-\left(\frac{\pi n\alpha}{l}\right)^2 t^\alpha\right)\sin\frac{\pi n}{l}x.$$

Used literature:

1. Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and Applications of Fractional Differential Equations, Elsevier (2006).
2. R. Ashurov, Yu. Fayziev, “On the nolocal boundary value problems for time-fractional equations,” Fractal and Fractional, 6, 41 (2022).
3. A.V.Pskhu The Stankovich Integral transform and Its Applications, SpecialFunctions and Analysis of Differntial Equations, 2020, p.16.
4. Umarov S., Hahn M., Kobayashi K. Beyond the triangle: Browian motion, Itocalculas, and Fokker-Plank equation-fractional generalizations. World Scientifi. 2017.