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OPTIMAL QUADRATURE FORMULA WITH DERIVATIVE FOR APPROXIMATING INTEGRALS AND SOLVING INTEGRAL EQUATIONS IN SOBOLEV SPACE

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Abstract. As a result of extensive scientific research conducted on a global scale, the numerical solution of exact integrals in the problems of solar physics, modeling of synthesized holograms, mechanics of liquids and gases with high accuracy leads to the construction of optimal quadrature formulas. Usually, the use of simple interpolation quadrature formulas in solving such problems requires a large amount of computational work. The effectiveness of quadrature formulas is characterized by their degree of accracy and order of accuracy. In this work, the derivative optimal quadrature formula is constructed using the first and second order derivatives of the function at the nodes in the real-valued Sobolev space of the differentiable functions.

Keywords. Sobolev space, quadrature formulas, extremal function, norm of the error functional, optimal coefficients.

INTRODUCTION

Numerical analysis is a branch of mathematics and mechanics that deals with the development of effective methods for obtaining numerical solutions to complex problems. Due to the development of computing technology, numerical problem solving is developing. Currently, the construction of derivative optimal quadrature formulas is developing rapidly. They are used to approximate definite integrals to finite sums. The effetiveness of quadrature formulas is characterized by their degree of accuracy and order of accuracy. Such formulas can be obtained using variational principles. Construction of optimal quadrature formulas in classes of differentiable functions is one of the current problems of computational mathematics. The problem of optimization of quadrature formulas using the variational approach involves minimizing the norm of the error functional in the given space of functions [1,2]. Among the quadrature formulas of the following form

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N}\sum_{j=0}^{\alpha} K_{\beta j}\varphi^{(j)}(x_\beta)$$

S.A.Michelli [3] in $W_2^{(m)}$ space m it is shown that it is the best formula when there are odd natural numbers, A.A.Jensikbayev [4] showed that the Euler-Maklauren formula is optimal in $L_2^{(m)}(0,1)$ space, Kh.M. Shadimetov [5] at work for $\alpha=0$ optimal quadrature formula is constructed in $L_2^{(m)}(0,1)$ space and error functional norm calculated. In the work of B. Bojanov, P. Petrov [6] degree all algebraic to polynomials accurate has been the following Gaussian type formula is studied

$$\int_a^b f(t)dt \cong \sum_{k=1}^{n} a_k \frac{1}{h_k}\int_{x_k}^{x_k+h_k} f(t)dt$$

V.I.Danchenko, L.A.Semin [7] Chebyshev-Markov multiplication from zeros used without complicated rational of functions integrals for quadrature formulas constructed and of them rational of functions L_2 norms count for used. In the work of W.F.Ford [8] singular integrals approx count for derivative quadrature formulas received and of the Riemann Zeta function derivatives count for efficient algorithm given. In this work, the following derivative quadrature formula is constructed

$$\int_0^1 \varphi(x)dx \cong \sum_{\beta=0}^{N} C_0[\beta]\varphi[\beta] + \frac{h^2}{12}(\varphi'(0)-\varphi'(1)) + \sum_{\beta=0}^{N} C_1[\beta]\varphi'[\beta]$$

(1)

with error functional

$$\ell_N(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} C_0[\beta]\delta(x-[\beta]) + \frac{h^2}{12}(\delta'(x)-\delta'(x-1)) - \sum_{\beta=0}^{N} C_1[\beta]\delta'(x-[\beta])$$

(2)

where $\varepsilon_{[0,1]}(x)$ - is the characteristic function of the interval [0,1], $\delta(x)$ - Dirac's delta function, $C_1[\beta]$, $\beta = \overline{0,N}$ unknown coefficients of the formula (1), $[\beta] = h\beta$, $h = \frac{1}{N}$, N - natural number, $\varphi \in L_2^{(m)}(0,1)$. In space $L_2^{(m)}(0,1)$ of the function norm as below determined

$$\|\varphi\|_{L_2^{(m)}} = \left(\int_0^1 \left(\frac{d^m(\varphi)}{dx^m} \right)^2 dx \right)^{\frac{1}{2}}$$

Error functional $\ell_N(x)$ defined in the $L_2^{(m)*}(0,1)$ conjunction space, it satisfies the following conditions [1]

$$(\ell_N, x^\alpha) = 0, \quad \alpha = 0,1,2,\ldots,m-1 \qquad (3)$$

from the definition of the norm of a functional

$$\left\|\ell_N \mid L_2^{(m)*}(0,1)\right\| = \sup_{\|\varphi|L_2^{(m)}(0,1)\| \neq 0} \frac{|(\ell_N,\varphi)|}{\left\|\varphi \mid L_2^{(m)}(0,1)\right\|}.$$

We have the Cauchy-Schwarz inequality

$$|(\ell_N,\varphi)| \le \|\varphi \mid L_2^{(m)}(0,1)\| \cdot \|\ell_N \mid L_2^{(m)*}(0,1)\|$$

It can be seen from this inequality that the error of the quadrature formula (1) is from above $\ell_N(x)$ is evaluated by the norm of the error functional. We minimize the norm of the error function only by the coefficients when the nodes are fixed.

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$$\left\| \mathring{\ell}_N | L_2^{(m)*} \right\| = \inf_{C_1[\beta]} \left\| \ell_N | L_2^{(m)*} \right\| \qquad (4)$$

that satisfy relation (4) $C_1[\beta]$ are called optimal coefficients. Corresponding to it $\ell_N(x)$ is called the optimal error functional and $\ell_N(x)$ is determined [9-11].

Theorem. $L_2^{(m)}(0,1)$ in Sobolev space for $m \quad 3$ the coefficients of the optimal quadrature formula in the form (1) are follows

$$C_1[0] = h^3 \sum_{k=1}^{m-3} a_k \frac{q_k^N - q_k}{1 - q_k}, \quad \beta = 0, N,$$

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3} a_k \left(q_k^\beta + q_k^{N-\beta} \right), \quad \beta = \overline{1, N-1}$$

here a_k,

$$\sum_{k=1}^{m-3} a_k \sum_{i=1}^{\alpha} \frac{q_k^{N+i} + (-1)^{i+1} q_k}{(1-q_k)^{i+1}} \Delta^i 0^\alpha = \frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}, \quad \alpha = \overline{1, m-3}.$$

PRELIMINARIES

In this section, we present certain definitions and formulas that will be needed in our proof of the main results obtained. Suppose that φ and ψ are functions of real variables and are defined in Ў space.

Definition 1. Function $\varphi[\beta]$ is called function of discrete argument, if it is given on some set of integer values of β [1,2].

Definition 2. Inner product of two discrete $\varphi[\beta]$ and $\psi[\beta]$ functions are

$$[\varphi, \psi] = \sum_{\beta=-} \varphi[\beta] \, \psi[\beta]$$

Definition 3. $\varphi[\beta]$ and $\psi[\beta]$ the convolution of functions with discrete arguments is defined as follows

$$\varphi[\beta] * \psi[\beta] = \sum_{\gamma=-} \varphi[\gamma] \, \psi([\beta] - [\gamma]).$$

We use the following equations derived from the sum formula of geometric progression [12]

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$$\sum_{\gamma=0}^{n-1} q^{\gamma}\gamma^{k} = \frac{1}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^{i}\Delta^{i}0^{k} - \frac{q^{n}}{1-q}\sum_{i=0}^{k}\left(\frac{q}{1-q}\right)^{i}\Delta^{i}\gamma^{k}\Big|_{\gamma=n}$$
(5)

where $\Delta^{i}\gamma^{k}$ is the finite difference of order i of γ^{k}, q is ratio of a geometric progression, $q<1$, $\Delta^{i}0^{k} = \sum_{\ell=1}^{i}(-1)^{i-\ell}C_{i}^{\ell}\ell^{k}$, $\Delta^{\alpha}x^{k} = \sum_{p=0}^{k} C_{k}^{p}\Delta^{\alpha}0^{p}x^{k-p}$.

In calculating some sums [13]

$$\sum_{\gamma=0}^{\beta-1}\gamma^{k} = \sum_{j=1}^{k+1}\frac{k!B_{k+1-j}}{j!(k+1-j)!}\beta^{j},$$
(6)

where B_{k+1-j} are Bernoulli numbers.

PROOF OF THEOREM

For proof of Theorem, we need to solve the following problems in sequence:

- find an overview of the norm of the error functional;
- obtain a system of equations of the Wiener-Hopf type;
- find optimal coefficients by solving a system of equations.

We use the concept of an extremal function to find a general representation of the norm of the error functional. In $L_{2}^{(m)}(0,1)$ space, of the error functional $\ell_{N}(x)$ and the extremal function $U_{\ell}(x)$ have the forms [1]

$$U_{\ell}(x) = (-1)^{m}\ell_{N}(x) * G_{m}(x) + P_{m-1}(x),$$

where $G_{m}(x) = \frac{|x|^{2m-1}}{2\ (2m-1)!}$, $P_{m-1}(x)$ is a algebraic polynomial degree of $(m-1)$.

We find the minimum of the error functional $\ell_{N}(x)$ by $C_{1}[\beta], \beta = 0,1,...,N$ coefficients under conditions [3]. For this, using the method of Lagrange unknown multipliers to find the conditional extremum of multivariable functions, we get the following system of equations

$$\sum_{\gamma=0}^{N} C_{1}[\gamma]G_{m-2}([\beta]-[\gamma]) + P_{m-3}[\beta] = F_{m}[\beta], \quad \beta = \overline{0,N}$$
(7)

$$\sum_{\gamma=0}^{N} C_{1}[\gamma][\gamma]^{\alpha} = -\sum_{j=1}^{\alpha}\frac{\alpha!B_{\alpha+3-j}h^{\alpha+3-j}}{j!(\alpha+3-j)!}, \quad \alpha = \overline{0,m-3}$$
(8)

here

$$f_m[\beta]=\sum_{i=0}^{2m-5}\frac{[\beta]^{2m-5-i}}{(2m-5-i)!}\ -\frac{B_{i+3}h^{i+3}}{(i+3)!}+\sum_{j=1}^{i}\frac{(-1)^i B_{i+3-j}h^{i+3-j}}{2j!(i+3-j)!}\ +\frac{B_{2m-2}h^{2m-2}}{(2m-2)!}$$

(9)

(7)-(8) are system of discrete equations of the Wiener-Hopf type. This system has a unique solution, and this solution gives a minimum value to $\|l_N\|^2$ (the square of the norm of the error function) [5]. To solve this system, we use the approach based on the d^{2m-2}/dx^{2m-2} discrete analogue of the $D_{m-2}[\beta]$ differential operator [14]. We rewrite the equation (7) in the form of a convolution, taking $C_1[\beta]=0$ for $\beta<0$ and $\beta>N$

$$G_{m-2}[\beta]*C_1[\beta]+P_{m-3}[\beta]=F_m[\beta],\quad \beta=0,1,...,N$$

(10)

and following substitution

$$v[\beta]=G_{m-2}[\beta]*C_1[\beta];\quad u[\beta]=v[\beta]+P_{m-3}[\beta]$$

$C_1[\beta]$ coefficients can be found by applying $D_{m-2}[\beta]$ discrete analogues to both sides of the equation (10)

$$C_1[\beta]=hD_{m-2}[\beta]*u[\beta].$$

(11)

To calculate the convolution of (11), we need to find the representation of the function $u[\beta]$ for all integer values of β.

$\beta<0$

$$v[\beta]=\sum_{i=0}^{m-3}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\ i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}-\sum_{i=m-2}^{2m-5}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\ i!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^i,$$

(12)

$\beta>N$

$$v[\beta]=-\sum_{i=0}^{m-3}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\ i!}\sum_{j=1}^{i}\frac{i!B_{i+3-j}h^{i+3-j}}{j!(i+3-j)!}+\sum_{i=m-2}^{2m-5}\frac{[\beta]^{2m-5-i}(-1)^i}{2(2m-5-i)!\ i!}\sum_{\gamma=0}^{N}C_1[\gamma][\gamma]^i$$

(13)

We find $C_1[\beta]$ optimal coefficients when $\beta=1,2,...,N-1$ using the exact form of functions with discrete arguments $D_{m-2}[\beta]$ and $u[\beta]$. We introduce the following designations

$$a_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6} q_k E_{2m-5}(q_k)} \sum_{\gamma=1} q_k^{\gamma} \left(R_{2m-5}[-\gamma] + Q_{m-3}^{-}[-\gamma] - F_m[-\gamma] \right), \quad (14)$$

$$b_k = \frac{(2m-5)!(1-q_k)^{2m-3}}{h^{2m-6} q_k E_{2m-5}(q_k)} \sum_{\gamma=1} q_k^{\gamma} \left(-R_{2m-5}(1+[\gamma]) + Q_{m-3}^{+}(1+[\gamma]) - F_m(1+[\gamma]) \right). \quad (15)$$

$C_1[\beta]$, $\beta = 1,2,...,N-1$ coefficients of the optimal quadrature formula of the form (1) for $m \geq 3$ in $L_2^{(m)}(0,1)$ space are as follows

$$C_1[\beta] = h^3 \sum_{k=1}^{m-3} \left(a_k q_k^{\beta} + b_k q_k^{N-\beta} \right), \quad \beta = 1,2,...,N-1$$

where a_k, b_k are determined by (14)-(15).

Therefore, $C_1[\beta]$ $\beta = \overline{1, N-1}$ optimal coefficients depend on a_k and b_k unknowns, and $2m-6$ equations are needed to determine them. First, we find the expression of coefficients $C_1[0]$ and $C_1[N]$ from equation (8)

$$C_1[0] = \sum_{\beta=1}^{N-1} C_1[\beta][\beta] - \sum_{\beta=1}^{N-1} C_1[\beta],$$

$$C_1[N] = -\sum_{\beta=1}^{N-1} C_1[\beta][\beta].$$

We calculate the following sum from (8)

$$S = \frac{[\beta]^{2m-5}}{(2m-5)!} C_1[0] - \sum_{i=0}^{2m-5} \frac{[\beta]^{2m-5-i} h^{i+3}}{(2m-5-i)!\, i!} \sum_{k=1}^{m-3} a_k \frac{q_k}{q_k - 1} \sum_{\alpha=0}^{i} \left(\frac{1}{q_k - 1} \right)^{\alpha} \Delta^{\alpha} 0^i$$

$$+ \sum_{k=1}^{m-3} p_k \frac{q_k^N}{1-q_k} \sum_{\alpha=0}^{i} \left(\frac{q_k}{1-q_k} \right)^{\alpha} \Delta^{\alpha} 0^i + \sum_{i=0}^{m-3} \frac{[\beta]^{2m-5-i} (-1)^i}{2(2m-5-i)!\, i!} \sum_{j=1}^{i} \frac{i! B_{i+3-j} h^{i+3-j}}{j!(i+3-j)!} \quad (16)$$

$$- \sum_{i=m-2}^{2m-5} \frac{[\beta]^{2m-5-i} (-1)^i}{2(2m-5-i)!\, i!} \sum_{\gamma=0}^{N} C_1[\gamma][\gamma]^i.$$

Substituting the expression (16) into (9), we equate the corresponding degrees of $[\beta]$

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$$C_1[0] = h^3 \sum_{k=1}^{m-3} \frac{a_k q_k^N - b_k q_k}{1-q_k}, \tag{17}$$

$$\sum_{k=1}^{m-3} \sum_{i=1}^{\alpha} \frac{a_k q_k + b_k q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^\alpha = \frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)}. \tag{18}$$

So, we solved this system of discrete equations (18) and got a system of $m-3$ equations to find the unknowns a_k and b_k. For $\alpha = 0$ in equation (8) using (16) and (17) we find $C_1[N]$

$$C_1[N] = h^3 \sum_{k=1}^{m-3} \frac{a_k q_k^N - b_k q_k}{1-q_k}.$$

We write the remainder of (8)

$$\sum_{\beta=0}^{N} C_1[\beta][\beta]^\alpha = -\sum_{j=1}^{\alpha} \frac{\alpha! B_{\alpha+3-j} h^{\alpha+3-j}}{j!(\alpha+3-j)!}, \quad \alpha = \overline{1,m-3}, \tag{19}$$

using (5) and (6) for the left side of (19), we get the following equality

$$\sum_{\beta=0}^{N} C_1[\beta][\beta]^\alpha = h^{\alpha+3} \sum_{k=1}^{m-2} \sum_{i=0}^{\alpha} \frac{a_k q_k^i + b_k q_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}} \Delta^i 0^\alpha$$

$$-\sum_{t=0}^{\alpha} \frac{\alpha! h^{t+3}}{t!(\alpha-t)!} \sum_{k=1}^{m-3} \sum_{i=0}^{\alpha} \frac{a_k q_k^{N+i} + (-1)^{i+1} b_k q_k}{(1-q_k)^{i+1}} \Delta^i 0^t + C_1[N]. \tag{20}$$

Substituting (20) into $C_1[N]$ we equate the corresponding degreess of h

$$\sum_{k=1}^{m-3} \sum_{i=0}^{\alpha} \frac{a_k q_k^{N+i} + (-1)^{i+1} b_k q_k}{(1-q_k)^{i+1}} \Delta^i 0^\alpha = \sum_{k=1}^{m-3} \sum_{i=0}^{\alpha} \frac{a_k q_k^{i} + (-1)^{i+1} b_k q_k^{N+1}}{(1-q_k)^{i+1}} \Delta^i 0^\alpha, \alpha = \overline{1,m-3} \tag{21}$$

$$\sum_{k=1}^{m-3} \sum_{i=0}^{j} \frac{a_k q_k^{N+i} + (-1)^{i+1} b_k q_k}{(1-q_k)^{i+1}} \Delta^i 0^j = \frac{B_{j+3}}{(j+1)(j+2)(j+3)}, \; j = \overline{1,\alpha-1}, \alpha = \overline{1,m-3} \tag{22}$$

From (18), (21) and (22) we get the following new system of equations for a_k and b_k unknowns

$$\sum_{k=1}^{m-3} \sum_{i=0}^{\alpha} \frac{a_k q_k^{N+i} + (-1)^{i+1} b_k q_k}{(1-q_k)^{i+1}} \Delta^i 0^\alpha = (-1)^{\alpha+1} \frac{B_{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} = 0, \;\; \alpha = \overline{1,m-3}$$

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Taking into account that $B_{2N+1} = 0$, subtracting (22) from (18) gives $a_k = b_k$.

NUMERICAL RESULTS

We numerically analyze the analytical results obtained in this section and compare them with other works. In $L_2^{(m)}(0,1)$ Sobolev space using functions $\varphi(x) = \cos 2x + 2e^{2x}$ and $f(x) = \ln(2+3x) - \frac{1}{2}e^{-2x}$, when $N = 10;50;100;150$, we analyze the quadrature formula's error (**Error 1**) constructed in work [15] and the quadrature formula's error (**Error 2**) constructed in this work in the tables below.

TABLE 1. Error between the exact and numerical solution of $\varphi(x)$

N	Exact solution	Error 1	Error 2
10	3.14569871	$1.7523*10^{-8}$	$1.04567*10^{-8}$
50	3.14569871	$2.23567*10^{-10}$	$2.45871*10^{-12}$
100	3.14569871	$1.45792*10^{-12}$	$1.56217*10^{-13}$
150	3.14569871	$1.75632*10^{-13}$	$2.54212*10^{-14}$

TABLE 2. Error between the exact and numerical solution of $f(x)$

N	Exact solution	Error 1	Error 2
10	-0.36412545	$4.57812*10^{-8}$	$1.45361*10^{-8}$
50	-0.36412545	$3.86957*10^{-11}$	$2.78957*10^{-12}$
100	-0.36412545	$4.27852*10^{-12}$	$1.78771*10^{-13}$
150	-0.36412545	$2.36364*10^{-13}$	$6.25657*10^{-14}$

It is known that finding an analytical solution of integral equations is quite complicated. If the kernel of the integral equation consists of complex functions, solving the integral equation analytically requires a lot of work. In Table 3, the numerical solution of the Fredholm integral equation of the second kind error is analyzed using Simpson formula (Error Simp) and optimal quadrature formula (Error OQF) with $N = 6$ nodes.

$$u(x) = e^x + 2\int_0^1 e^{x+t}u(t)dt$$

TABLE 3. Numerical solution of integral equation

N	Exact solution	Error Simp	Error OQF
0	-0.1855612526	$1.485665 *10^{-5}$	$1.332376 *10^{-6}$

1/6	-0.2192147180	$1.754528 *10^{-5}$	$1.584508 *10^{-6}$
1/3	-0.2589715897	$1.074037 *10^{-5}$	$1.861457 *10^{-6}$
1/2	-0.3059387842	$1.454573 *10^{-5}$	$1.202478 *10^{-6}$
2/3	-0.3614239684	$1.904037 *10^{-5}$	$1.607452 *10^{-6}$
5/6	-0.4269719685	$1.424573 *10^{-5}$	$1.072541 *10^{-6}$
1	-0.5044077809	$1.044531 *10^{-5}$	$1.452323 *10^{-6}$

CONCLUSION

In this research work, we have constructed derivative optimal quadrature formulas with derivative for the approximation of definite integrals and the numerical solution of integral equations. Using the extremal function, we obtained an overview of the norm of the error function. Since the error function is a multivariate function, we found its extremum under certain conditions. By solving the system of equations, we found the analytical representation of the coefficients. We numerically analyzed that the obtained results are more effective than classical formulas.

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