SOLVING FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS IN HEAT CONDUCTION AND BIOLOGICAL MODELING

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Annotation: This article analyzes the method of solving fractional integro-differential equations using the Neumann series. The convergence of the solution for equations based on the Caputo fractional derivative is proven by applying the Banach fixed-point theorem. The efficiency of the method is demonstrated through numerical examples and is applied to problems of heat conduction and biological population dynamics. The results are of significant importance in modern materials science and biology, and contribute to the advancement of fractional mathematics within the scientific community of Uzbekistan.
Keywords: fractional integro-differential equations, Neumann series, Caputo derivative, convergence, heat conduction, biological modeling.

Introduction. In recent years, fractional differential and integro-differential equations have taken on an important role in mathematics and applied sciences [1, 2]. They are used to model memory-effect processes, such as anomalous diffusion, viscoelastic materials, biological population dynamics, and signal transmission [3, 4]. In Uzbekistan's mathematical school, research on fractional equations is evolving, particularly through the works of Kadirkulov and Khudaybergenov [5].
Compared to classical differential equations, fractional equations provide a more accurate description of the behavior of complex systems. However, obtaining analytical solutions for such equations is challenging; hence, iterative methods, especially the Neumann series, are widely employed [6]. The Neumann series is an effective tool for solving integral equations, and its convergence is justified through the Banach fixed-point theorem [7].
This article examines the method of solving fractional integro-differential equations via the Neumann series. The convergence of the solution is analyzed, and the method is applied to heat conduction and biological population dynamics problems. The aim of this paper is to demonstrate the effectiveness of the method, verify it through numerical simulations, and contribute to the development of fractional mathematics in the scientific context of Uzbekistan.
Definition 1. The Caputo fractional derivative of order α\alphaα is defined as follows:

$$D^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-s)^{-\alpha}u'(s)ds$$

where $\Gamma(z)=\int_{0} e^{-t}t^{z-1}dt$ is the gamma function [1].

Definition 2. An integral equation of the form

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$$u(t) = f(t) + \int_0^t K(t,s)D^{\alpha}u(s)ds \ , \ t \in [0,T]$$

in which the upper limit of integration depends on a free variable, is called a Volterra integral equation of the second kind. Here, $u(t)$ is the unknown function to be determined, $f(t)$ is a given function, $K(t,s)$ is the kernel function ,where $f(t) \in C[0,T], K(t,s) \in C([0,T] \times [0,T])$.

Definition 3. The Neumann series is an iterative method for solving integral equations and is defined as:

$$u_0(t) = f(t) \ , \ u_{n+1}(t) = f(t) + \int_0^t K(t,s)D^{\alpha}u_n(s)ds \ , \ n = 0,1,2,.......$$

If the convergence conditions are satisfied, the solution can be expressed as:

$$u(t) = \lim_{n\to\infty} u_n(t) \qquad [7].$$

Theorem 1. (Banach Fixed-Point Theorem). If the operator

$$A : u \to f(t) + \int_0^t K(t,s)D^{\alpha}(s)ds$$

is a contraction, that is, $\|Au - Av\| \leq q\|u - v\|, \ q < 1$ hen the equation has a unique solution, and the Neumann series converges to this solution [8].

Now, let us consider the application of the Neumann series to fractional integro-differential equations and analyze the convergence of the solution.

Let us examine the following equation:

$$u(t) = f(t) + \int_0^t K(t,s)D^{\alpha}u(s)ds \ , t \in [0,T]$$

here, $f(t) \in C[0,T], K(t,s) \in C([0,T] \times [0,T])$ and D^{α} is the Caputo fractional derivative.

The operator is defined as follows:

$$Au(t) = f(t) + \int_0^t K(t,s)D^{\alpha}u(s)ds$$

The Neumann series iterations are given by: $u_0(t) = f(t)$,

$$u_{n+1}(t) = f(t) + \int_0^t K(t,s)D^{\alpha}u_n(s)ds$$

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To demonstrate that the operator A is contractive, we verify the following condition:

$$\left\| Au - Av \right\| = \left\| \int_0^t K(t,s) - D^{\alpha}(u(s) - v(s))ds \right\|$$

Regarding the properties of the Caputo derivative:

$$\left\| D^{\alpha}(u - v) \right\| \quad \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \left\| u - v \right\|$$

If $\sup_{t \ [0,T]} \int_0^t |K(t,s)| ds \quad K_0$ then $\left\| Au - Av \right\| \quad K_0 \quad \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \left\| u - v \right\| = q \left\| u - v \right\|$,

where $q = K_0 \quad \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$ .Therefore, the operator A is a contraction, and by Banach's fixed-point theorem, the Neumann series converges to a unique solution.

Example. Let us consider the equation for the parameters $f(t) = t^2$, $K(t,s) = e^{s-t}$, $\alpha = 0,5, T = 1$.

$$u(t) = t^2 + \int_0^t e^{s-t} D^{0,5} u(s) ds$$

tamiz First, let us examine the first two iterations of the Neumann series..

$u_0(t) = t^2$, $u_1(t) = t^2 + \int_0^t e^{s-t} D^{0,5}(s^2) ds$, $D^{0,5}(s^2) = \frac{1}{\Gamma(0,5)} \int_0^s (s-\tau)^{-0,5} 2\tau d\tau = \frac{4\sqrt{s^3}}{3\sqrt{\pi}}$

$$u_1(t) = t^2 + \int_0^t e^{s-t} \frac{4\sqrt{s^3}}{3\sqrt{\pi}} ds$$

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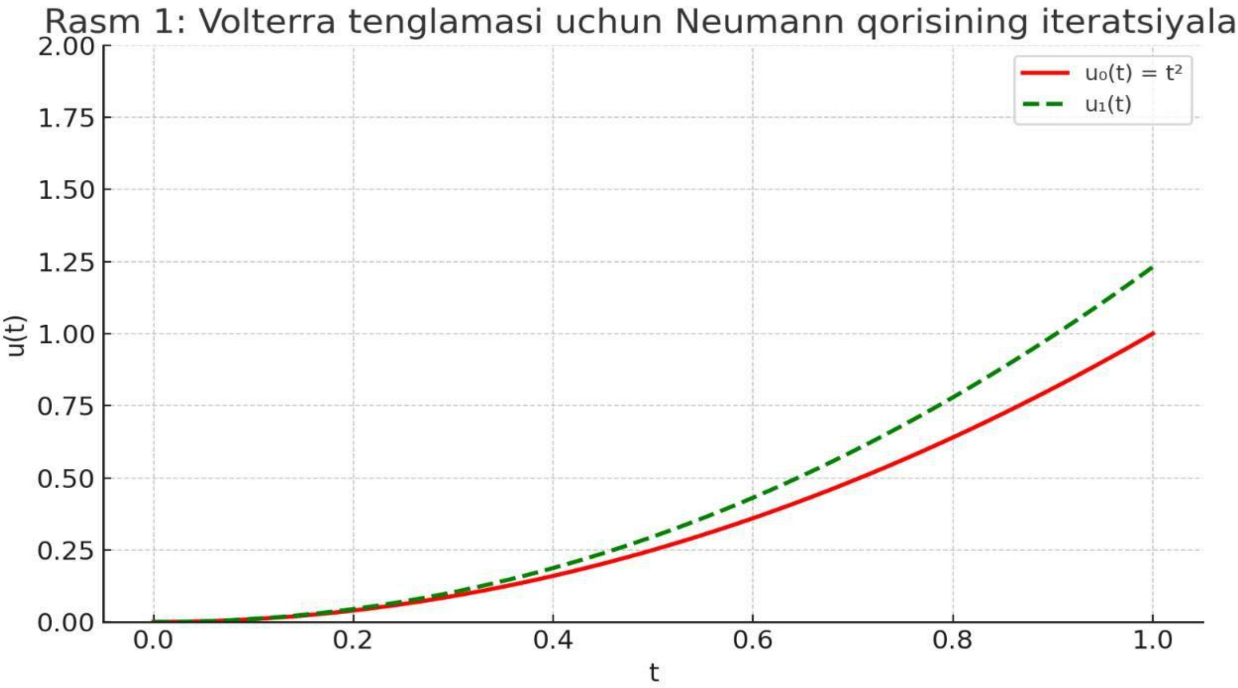

The convergence of the iterated kernels mentioned above can be visualized by generating their graphs using Python.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad
from scipy.special import gamma
```

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```
T = 1; alpha = 0.5; n = 100
t = np.linspace(0, T, n)
f = lambda t: t**2
K = lambda t, s: np.exp(s - t)
Gamma = gamma(1.5)
u0 = f(t)
u1 = np.zeros(n)
for i in range(n):
integrand = lambda s: K(t[i], s) * (4 * np.sqrt(s**3) / (3 * np.sqrt(np.pi))) # D^0.5 t^2
u1[i] = f(t[i]) + quad(integrand, 0, t[i])[0]
plt.plot(t, u0, 'b-', label='u_0(t) = t^2', linewidth=2)
plt.plot(t, u1, 'r--', label='u_1(t)', linewidth=2)
plt.grid(True)
plt.xlabel('t')
plt.ylabel('u(t)')
plt.title('Rasm 1: Volterra tenglamasi uchun Neumann qorisining iteratsiyalari')
plt.legend()
plt.savefig('rasm1.png', dpi=300)
plt.savefig('rasm1.pdf')
plt.show()
```

Now, let us examine the practical application of the Neumann series. Fractional integro-differential equations are applied in two important fields: **heat conduction** and **biological modeling**.

The fractional heat equation is expressed as follows:

$$D^{\alpha}u(t,x) = k\frac{\partial^2 u}{\partial x^2},\ 0 < \alpha < 1$$

here $u(x,t)$ represents the temperature distribution, k - is the thermal conductivity coefficient. This equation is transformed into an integral form and the Neumann series is applied to obtain the solution:

$u(t,x) = u_0 + \int_0^t G(t-s,x)D^{\alpha}u(s,x)ds$ bu yerda $G(t,x)$ -Grin funksiyasi.

The following figure illustrates $\alpha = 0,7$ how anomalous diffusion behaves differently from classical diffusion.

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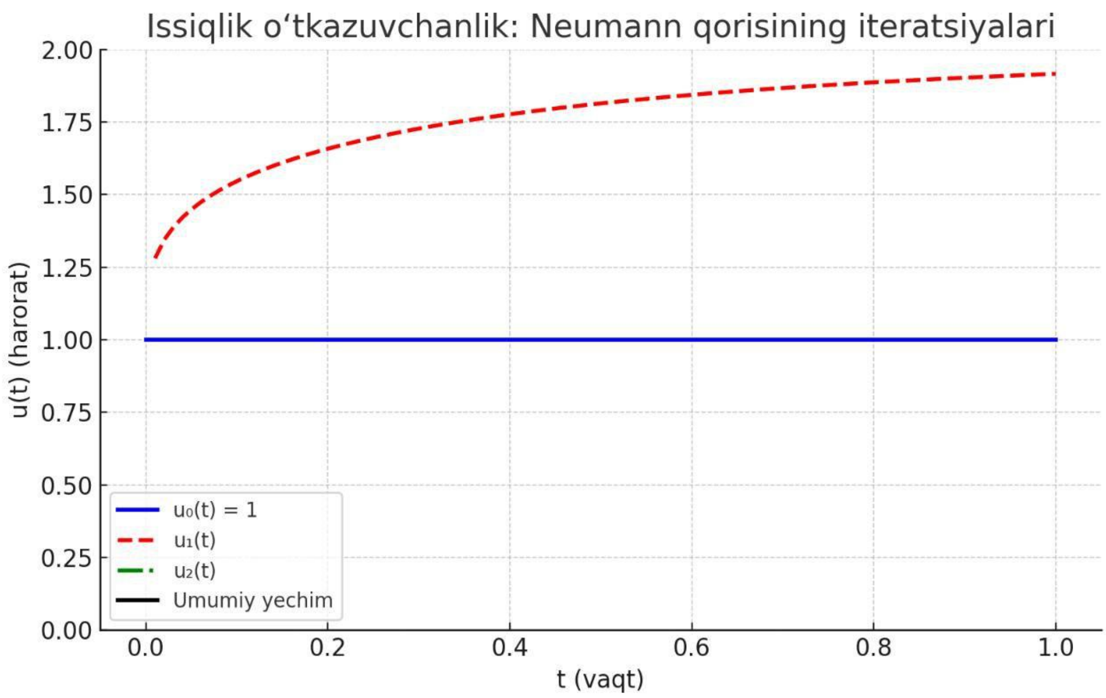


```
plt.plot(t, u[0], 'b-', label='u_0(t) = 1', linewidth=2)  # Boshlang'ich taxmin
plt.plot(t, u[1], 'r--', label='u_1(t)', linewidth=2)  # Birinchi iteratsiya
plt.plot(t, u[2], 'g-.', label='u_2(t)', linewidth=2)  # Ikkinchi iteratsiya
plt.plot(t, u[-1], 'k-', label='Umumiy yechim (sonli yaqinlash)', linewidth=2)
plt.grid(True)
plt.xlabel('t (vaqt)')
plt.ylabel('u(t) (harorat)')
plt.title(Neumann qorisining 3 iteratsiyasi va umumiy yechim')
plt.legend()
plt.ylim(0, 2)
plt.savefig('issiqlik_grafik.png', dpi=300)
plt.savefig('issiqlik_grafik.pdf')
plt.show()
```

```
n = 100  # Nuqtalar soni
t = np.linspace(0, T, n)
k = 1  # Diffuziya koeffitsienti (o'zgartirish mumkin)
u0 = lambda x: np.ones_like(x)  # Boshlang'ich shart: u0(x) = 1
G = lambda t_s, x: np.exp(-(t_s))  # Yadro funksiyasi: e^-(t-s)
Gamma = gamma(1 - alpha)  # Gamma(0.3) uchun koefitsient
u = [u0(t)]  # u0(t) = 1
for k_iter in range(10):  # 10 iteratsiyagacha hisoblash
    u_new = np.zeros(n)
    for i in range(n):
```

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more complex systems, such as multi-dimensional fractional equations or real-time computational algorithms.

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